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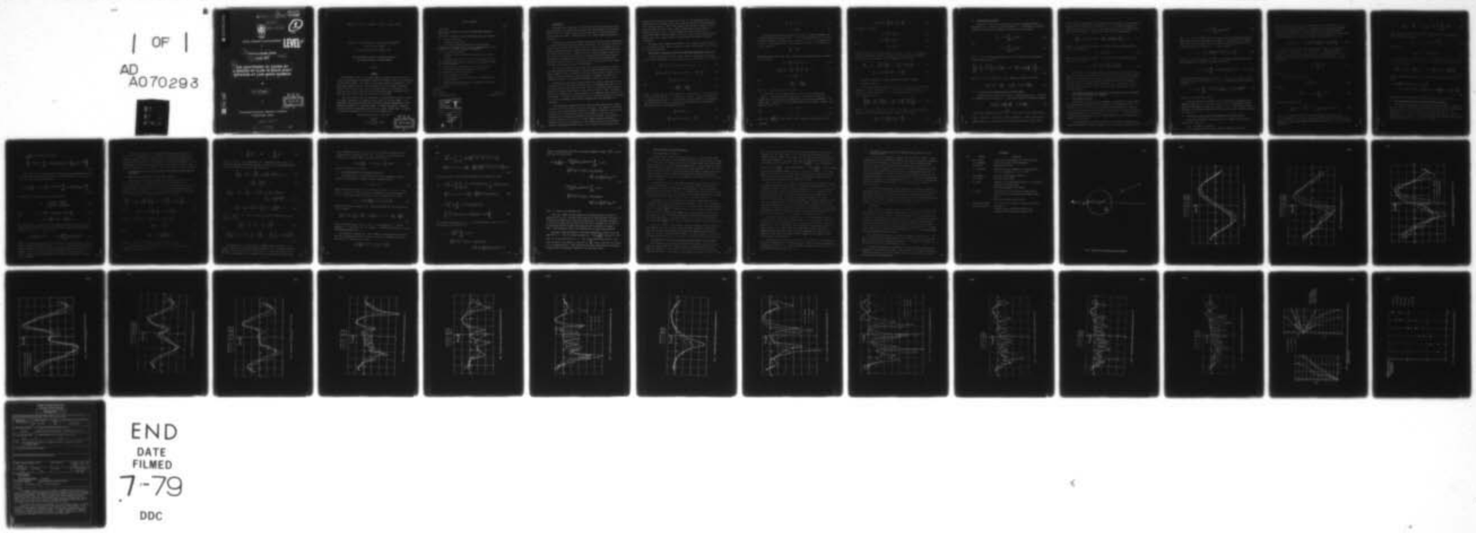

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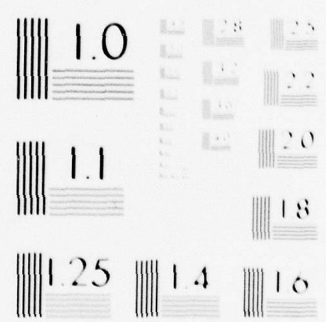

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ROYAL AIRCRAFT ESTABLISHMENT

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(9) Technical Report 78100

(11) August 1978

(12) 41p.

(6) THE SCATTERING OF SOUND BY A REGION OF FLUID IN SOLID BODY ROTATION AT LOW MACH NUMBER.

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UDC 534.5 : 532.527 : 533.696.5 : 534.25 : 534-14 : 534.26

ROYAL AIRCRAFT ESTABLISHMENT

Technical Report 78100

Received for printing 15 August 1978

THE SCATTERING OF SOUND BY A REGION OF FLUID IN SOLID BODY ROTATION AT LOW MACH NUMBER

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SUMMARY

This example, which is one of the simplest to exhibit vortex refraction of sound, is solved using an approximation which is uniformly valid at all frequencies and small Mach numbers. The region of solid body rotation is bounded by a vortex sheet which implies that the results obtained have little physical relevance. However this flow field gives acoustic equations which are comparatively easy to solve and thus an ideal test case for the various approximate techniques used to investigate the effect of more realistic vortices on sound.

The present Report gives an assessment of a method which sets out to express the acoustic pressure as an asymptotic power series in the Mach number. It is found that this approach produces an error which increases progressively as the frequency increases at a fixed Mach number. At higher frequencies it appears that the ray theory approximation is likely to be more accurate and it is hoped to publish an assessment of the utility of this method later.

Departmental Reference: Aero 3440

TR 78100

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I INTRODUCTION

Experiments have shown that the refraction caused by vortices shed from lifting surfaces can significantly influence the noise nuisance of an aircraft[1]. This fact has led to vortex refraction being the subject of many theoretical investigations, in order to understand the effect better, and even perhaps to exploit it.

The majority of published analyses are based on ray theory whose application to vortex flow has been well described by Broadbent[2]. Ray theory has two built-in restrictions which may be violated in practice. First, it is assumed that the acoustic wavelength is small compared with the length of significant variation of the underlying flow field. Secondly the results obtained cannot be applied in the limit of the extreme far field if the predicted acoustic pressure varies by large amounts in different directions due to such effects as wing shielding. Ray theory takes no account of the smearing out of discontinuities like shadow boundaries which takes place over large distances.

The first restriction is not serious since ray theory has been shown to give good agreement with the results from wind-tunnel experiments where microphones normally cannot be placed very far away from the model. However, if one wishes to predict an aircraft noise footprint, for example, then most of the regions of interest on the ground will probably be well into the far field.

Broadbent also mentioned a second theoretical approach, which was tried as long ago as 1959[6], based on the assumption that the maximum Mach number M of the vortex flow is small compared to one. This is a reasonable approximation to make for an aircraft which has just taken off or is about to land. The method consists of expressing the acoustic pressure as an asymptotic power series in the Mach number. In practice this process has so far only been applied to first order in M, since the higher order terms get progressively more difficult to calculate.

Both theoretical approaches, which have just been outlined, pose various questions as to their range of practical application. The purpose of analysing in detail the particular example described in section 4 (which is not physically realistic, but is comparatively easy to solve) was to assess the errors introduced by the different approximations that can be made. The analysis is greatly simplified by the use of a low Mach number approximation which is uniformly valid for all frequencies. This contrasts with the last described method of approximation which clearly, from the form of the equations to be

solved and the results presented in this Report, is not uniformly valid as the frequency tends to infinity. The results from the uniformly valid approximation will contain an error of $O(M^2)$ when compared with the exact solution, but the author believes that this error is always insignificant for the range of M considered (from 0 to 0.3). Thus the results described in section 6, for various Mach numbers and wave numbers, serve as a check on any technique used to analyse the effect of more realistic low Mach number flow fields on acoustic propagation.

This Report only contains an assessment of the asymptotic power series in M for acoustic pressure, but it is hoped to produce an assessment of ray theory in a future publication.

2 THE ACOUSTIC EQUATIONS IN STEADY TWO-DIMENSIONAL HOMENTROPIC VORTEX FLOW

In what follows, the adjective 'basic' refers to the flow in the absence of any acoustic disturbance. The acoustic equations were derived in Ref 4 and are

$$\frac{\partial \rho}{\partial t} + \vec{Q} \, . \, \nabla \rho + \vec{q} \, . \, \nabla R + R \, \mathrm{div} \, \vec{q} + \rho \, \mathrm{div} \, \vec{Q} = 0 \quad , \tag{1}$$

$$\frac{\partial \vec{q}}{\partial t} + \mathrm{curl} \, \vec{Q} \times \vec{q} + \mathrm{curl} \, \vec{q} \times \vec{Q} + \nabla \vec{Q} \, . \, \vec{q} = - \frac{\nabla p}{R} + \frac{\rho \nabla P}{R^2} \quad , \tag{2}$$

$$\frac{\partial s}{\partial t} + \vec{Q} \, . \, \nabla s + \vec{q} \, . \, \nabla S = 0 \quad , \tag{3}$$

and

$$p = \left(\frac{\partial P}{\partial R}\right)_S \rho + \left(\frac{\partial P}{\partial S}\right)_R s \quad , \tag{4}$$

where t is the time, R basic density, $\vec{Q}$ basic velocity, P basic pressure, S basic specific entropy, ρ is the acoustic density, $\vec{q}$ acoustic velocity, p acoustic pressure, and s is the acoustic specific entropy. If viscous and heat conductivity effects are ignored, the usual equations of fluid mechanics show that the basic state quantities satisfy

$$\frac{\partial R}{\partial t} + \mathrm{div} \, R\vec{Q} = 0 \quad , \tag{5}$$

$$\frac{\partial \vec{Q}}{\partial t} + \mathrm{curl} \, \vec{Q} \times \vec{Q} + \nabla \tfrac{1}{2} Q^2 = - \frac{\nabla P}{R} \quad . \tag{6}$$

$$\frac{\partial S}{\partial t} + \vec{Q} \cdot \nabla S = 0 \quad , \tag{7}$$

and

$$P = P(R,S) \quad . \tag{8}$$

We will now particularise equations (1) to (8) to acoustic propagation in a steady two-dimensional vortex flow, *ie* $\vec{Q}$ has the form $(0,V(r),0)$ in a system of cylindrical polar coordinates (r,θ,z), and R, P and S are independent of z and t. Then equations (5) to (8) show that R, P and S are functions of r only, and that

$$\frac{dP}{dr} = \frac{RV^2}{r} \quad . \tag{9}$$

Substituting these particular forms of the basic state quantities into equations (1) to (4) gives

$$D\rho + u\frac{dR}{dr} + R \operatorname{div} \vec{q} = 0 \quad , \tag{10}$$

$$D\vec{q} + \left(-\frac{2Vv}{r}, \frac{u}{r}\frac{drV}{dr}, 0\right) + \frac{\nabla p}{R} - \frac{\rho \nabla P}{R^2} = 0 \quad , \tag{11}$$

$$Ds + u\frac{dS}{dr} = 0 \quad , \tag{12}$$

and

$$p = \left(\frac{\partial P}{\partial R}\right)_S \rho + \left(\frac{\partial P}{\partial S}\right)_R s \quad . \tag{13}$$

where $D \equiv \frac{\partial}{\partial t} + \frac{V}{r}\frac{\partial}{\partial \theta}$ and $\vec{q} = (u,v,w)$.

Finally we add the assumption that the basic state is homentropic, *ie* $\frac{dS}{dr} = 0$. Then equation (12) shows that s is constant for each fluid particle for all time and thus is independent of the acoustic disturbance. This implies that s must be zero for the homentropic flow considered. Further, from equations (8) and (13),

$$\nabla P = C^2 \nabla R \tag{14}$$

and

$$p = C^2 \rho \quad , \tag{15}$$

where $C(r) \equiv \sqrt{\left(\frac{\partial P}{\partial R}\right)_S}$ is the speed of sound. Equation (10) multiplied by $\frac{C^2}{R}$ now gives

$$D\frac{p}{R} + u\frac{V^2}{r} + C^2\left(\frac{1}{r}\frac{\partial ru}{\partial r} + \frac{\partial v}{r\partial\theta} + \frac{\partial w}{\partial z}\right) = 0 \quad , \tag{16}$$

and equation (11) implies

$$Du - \frac{2Vv}{r} + \frac{\partial}{\partial r}\frac{p}{R} = 0 \quad , \tag{17}$$

$$Dv + \frac{u}{r}\frac{drV}{dr} + \frac{\partial}{r\partial\theta}\frac{p}{R} = 0 \quad , \tag{18}$$

and

$$Dw + \frac{\partial}{\partial z}\frac{p}{R} = 0 \quad . \tag{19}$$

Equations (18) and (19) may now be used to eliminate v and w from equations (16) and (17) and thus obtain the following two equations which conveniently describe acoustic propagation in steady homentropic vortex flow:

$$\left(\frac{D^2}{C^2} - \frac{\partial^2}{r^2\partial\theta^2} - \frac{\partial^2}{\partial z^2}\right)\frac{p}{R} + \left\{D\left(\frac{\partial}{\partial r} + \frac{1}{r} + \frac{V^2}{C^2 r}\right) - \frac{1}{r}\frac{drV}{dr}\frac{\partial}{r\partial\theta}\right\}u = 0 \tag{20}$$

and

$$\left(D\frac{\partial}{\partial r} + \frac{2V}{r}\frac{\partial}{r\partial\theta}\right)\frac{p}{R} + \left(D^2 + \frac{2V}{r^2}\frac{drV}{dr}\right)u = 0 \quad . \tag{21}$$

3 THE ACOUSTIC EQUATIONS IN FLUID UNDERGOING SOLID BODY ROTATION

3.1 The exact equations

Examination of equations (20) and (21) shows that they have their simplest form when V varies linearly with r. For then D and the operator on u in equation (21) become independent of r, and equations (20) and (21) can be easily combined to give

$$\left[D^2\left(\frac{D^2}{C^2} - V^2\right) + \frac{V_0^2 D}{C^2 a^2}\left\{D\left(4 - r\frac{\partial}{\partial r}\right) - \frac{2V_0}{a}\frac{\partial}{\partial\theta}\right\} - \frac{4V_0^2}{a^2}\frac{\partial^2}{\partial z^2}\right]\frac{p}{R} = 0 \quad , \tag{22}$$

where D is now $\frac{\partial}{\partial t} + \frac{V_0}{a}\frac{\partial}{\partial\theta}$ and $V = \frac{V_0 r}{a}$, *ie* the basic velocity distribution takes the form for a solid body rotation. Equation (21) becomes

$$\left(D\frac{\partial}{\partial r} + \frac{2V_0}{a}\frac{\partial}{r\partial\theta}\right)\frac{p}{R} + \left(D^2 + \frac{4V_0^2}{a^2}\right)u = 0 \quad . \tag{23}$$

3.2 Separating the variables

From now on we will be dealing with an acoustic disturbance which is independent of z, and has a constant frequency ω. Accordingly, the acoustic pressure and radial velocity component can be written in the forms

$$\frac{p}{R} = e^{i\omega t} \sum_{n=-\infty}^{\infty} f_n(\xi) e^{in\theta} \quad , \tag{24}$$

and

$$u = e^{i\omega t} \sum_{n=-\infty}^{\infty} g_n(\xi) e^{in\theta} \quad , \tag{25}$$

where $\xi \equiv \frac{r}{a}$. Substituting expression (24) into equation (22), and equating Fourier components, gives

$$\left[\frac{d^2}{d\xi^2} + \left(1 + \frac{C_0^2 M^2 \xi^2}{C^2}\right)\frac{1}{\xi}\frac{d}{d\xi} + \frac{C_0^2}{C^2}\left\{(Ka + Mn)^2 - 2M^2\left(1 + \frac{Ka}{Ka + Mn}\right)\right\} - \frac{n^2}{\xi^2}\right] f_n = 0 \ , \qquad \ldots\ldots \ (26)$$

where $C_0 \equiv C(a)$, $M \equiv \frac{V_0}{C_0}$ and $K \equiv \frac{\omega}{C_0}$. Similarly, equation (23) gives

$$\left\{(Ka + Mn)^2 - 4M^2\right\} g_n = \frac{i}{C_0}\left\{(Ka + Mn)\frac{d}{d\xi} + \frac{2Mn}{\xi}\right\} f_n \quad . \tag{27}$$

3.3 A uniformly valid approximation at low Mach number

Equation (26) has two different types of term dependent on M. The first is contained in the variation of C, which can be quantified using the equation

$$\frac{1}{C_0}\frac{dC}{d\xi}\left(= \frac{1}{C_0}\left(\frac{\partial C}{\partial R}\right)_S \frac{dR}{d\xi}\right) = M^2 \frac{C_0}{C}\frac{R}{C}\left(\frac{\partial C}{\partial R}\right)_S \xi \quad , \tag{28}$$

and is also contained in the term $\frac{C_0^2 M^2 \xi^2}{C^2}$ which is part of the coefficient of $\frac{1}{\xi}\frac{d}{d\xi}$. Provided $\xi = O(1)$, these terms are uniformly of $O(M^2)$ and so may be ignored when $M \ll 1$, with an error in the final answer which is truly $O(M^2)$.

The second type of dependence is characterised by the parameter Mn which may or may not be small, depending on the value of n. It is therefore important to retain all the terms containing Mn if serious errors are to be avoided even at quite small Mach numbers. Equation (26), then, will be approximated by

$$\left\{\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} + (Ka + Mn)^2 - 2M^2\left(1 + \frac{Ka}{Ka + Mn}\right) - \frac{n^2}{\xi^2}\right\} f_n = 0 \qquad (29)$$

which is a form of Bessel's equation. The solution of this equation which is finite at the origin is

$$f_n = A_n J_n(\alpha_n \xi) \quad , \qquad (30)$$

where A_n is an arbitrary constant, J_n is the Bessel function of the first kind and nth order, and

$$\alpha_n \equiv \sqrt{(Ka + Mn)^2 - 2M^2\left(1 + \frac{Ka}{Ka + Mn}\right)} \quad . \qquad (31)$$

The advantage of employing the above approximation is clear in this case, since f_n can then be evaluated in terms of a function which is well documented. However, the author believes that the classification of the terms containing M into two types is important in general, and only terms which are uniformly small may safely be ignored without significant error. This point of view is reinforced by the results of section 5 where a cruder approximation process is used for comparison.

4 SCATTERING BY A REGION OF FLUID IN SOLID BODY ROTATION OF SOUND GENERATED BY A CONSTANT FREQUENCY LINE MONOPOLE

4.1 The mathematical model

The particular problem which we are going to investigate in detail, using the approximation developed in the last section, is illustrated in Fig 1. It consists of a region of fluid in solid body rotation bounded by the cylinder $\xi = 1$, with an infinite region of stagnant fluid outside containing a monopole of acoustic pressure on the line given by $\xi = \xi_0$ and $\theta = \pi$. The possibility of instabilities developing on the vortex sheet separating the two regions will be ignored in this analysis.

The approximation described in section 3.3 is uniformly valid since the region of solid body rotation is bounded. Thus the acoustic pressure is given by 100

$$p \;=\; \mathrm{Re}^{i\omega t} \sum_{n=-\infty}^{\infty} A_n J_n(\alpha_n \xi) e^{in\theta} \tag{32}$$

when $\xi \leqslant 1$. In the stagnant fluid, the acoustic pressure is the sum of two outgoing waves, one due to the monopole, the other to scattering by the rotating fluid. The first contribution is the incident wave which represents the total acoustic pressure in the limit $M \to 0$. This is given by

$$p_i \;\equiv\; P_i e^{i\omega t} H_0^{(2)}\left(Ka\sqrt{\xi^2 + \xi_0^2 + 2\xi\xi_0 \cos\theta}\right) \quad , \tag{33}$$

where P_i is a given constant and $H_n^{(2)}$ is the Hankel function of the second kind and nth order. Equation (33) can be rewritten as

$$p_i \;=\; P_i e^{i\omega t} \sum_{n-\infty}^{\infty} (-1)^n H_n^{(2)}(Ka\xi_0) J_n(Ka\xi) e^{in\theta} \qquad (\xi < \xi_0), \tag{34}$$

by using Graf's addition theorem. (See Ref 3, result 9.1.79.) Then the total acoustic pressure, in the region $1 \leqslant \xi \leqslant \xi_0$, is given by an infinite series of the form

$$p \;=\; P_i e^{i\omega t} \sum_{n=-\infty}^{\infty} (-1)^n H_n^{(2)}(Ka\xi_0) \left\{ J_n(Ka\xi) + B_n H_n^{(2)}(Ka\xi) \right\} e^{in\theta} \quad , \tag{35}$$

where B_n is an arbitrary constant.

Since the expansion in equation (34) is required for the application of boundary conditions at $\xi = 1$ so that A_n and B_n can be determined, we have chosen the form which is valid in the region $\xi < \xi_0$. This restriction does not affect the subsequent evaluation of the far-field acoustic pressure described in section 4.3.

4.2 Conditions on the vortex sheet bounding the solid body rotation

There are two boundary conditions which are normally applied across a vortex sheet. They are:

(i) continuity of pressure;

(ii) continuity of the velocity component normal to the sheet.

The first condition ensures that the acceleration of fluid particles in the sheet is finite, and the second is a consequence of the principle of conservation of mass. Assuming that the basic pressure has been made continuous, condition (i) implies that p is continuous across the sheet and, in particular, the Fourier components in equations (32) and (35) are equal at $\xi = 1$, *ie*

$$R(a)A_n J_n(\alpha_n) = P_i(-1)^n H_n^{(2)}(Ka\xi_0)\left\{J_n(Ka) + B_n H_n^{(2)}(Ka)\right\} . \tag{36}$$

Condition (ii) is more complicated to apply. Suppose the surface of the vortex sheet is described by $\Sigma = 0$. The form of Σ should show that the acoustic disturbance linearly perturbs the sheet away from the circle $r = a$ which the sheet would occupy if the line source were absent. Thus Σ may be defined by

$$\Sigma \equiv r - a - e^{i\omega t} \sum_{n=-\infty}^{\infty} \delta_n e^{in\theta} , \tag{37}$$

where $\delta_n \ll a$.

For a point $\vec{r}_\sigma$ on the sheet,

$$\frac{\partial \Sigma}{\partial t} + \frac{d\vec{r}_\sigma}{dt} . \nabla\Sigma = 0 , \tag{38}$$

and thus

$$\frac{d\vec{r}_\sigma}{dt} . \nabla\Sigma = i\omega e^{i\omega t} \sum_{n=-\infty}^{\infty} \delta_n e^{in\theta} . \tag{39}$$

Condition (ii) implies that

$$\lim_{\xi\to1+} \vec{q} . \nabla\Sigma = \frac{d\vec{r}_\sigma}{dt} . \nabla\Sigma = \lim_{\xi\to1-} \left\{(0,V_0,0) + \vec{q}\right\} . \nabla\Sigma , \tag{40}$$

where $\xi\to1+$ means that the unit circle is approached from outside, and $\xi\to1-$ means that the unit circle is approached from inside. After rejecting the products of linear perturbations and using the definition of Σ , these equations simplify to

$$\lim_{\xi \to 1+} u = \frac{d\vec{r}_\sigma}{dt} \cdot \nabla\Sigma = \lim_{\xi \to 1-} u - \frac{iV_0}{a} e^{i\omega t} \sum_{n=-\infty}^{\infty} n\delta_n e^{in\theta} \quad . \qquad (41)$$

Thus, combining equations (39) and (41), using equation (25), and equating Fourier components, we obtain

$$g_n(1+) = i\omega\delta_n = g_n(1-) - \frac{iV_0 n}{a} \delta_n \qquad (42)$$

which leads to

$$Kag_n(1-) = (Ka + Mn)g_n(1+) \quad . \qquad (43)$$

Now, equation (27) applies outside the solid body rotation if M is made zero, and so it can be used to express g_n in terms of f_n on both sides of equation (43) to give

$$K^2a^2\left\{(Ka + Mn)\frac{d}{d\xi} + 2Mn\right\}f_n(1-) = (Ka + Mn)\left\{(Ka + Mn)^2 - 4M^2\right\}\frac{df_n(1+)}{d\xi} \quad .$$

...... (44)

Finally, substituting the forms of f_n implied by equations (32) and (35), we obtain

$$R(a)A_nKa\left\{(Ka + Mn)\alpha_n J_n'(\alpha_n) + 2MnJ_n(\alpha_n)\right\}$$

$$= P_i(-1)^n H_n^{(2)}(Ka\xi_0)(Ka + Mn)\left\{(Ka + Mn)^2 - 4M^2\right\}\left\{J_n'(Ka) + B_nH_n^{(2)\prime}(Ka)\right\} \quad .$$

...... (45)

Equations (36) and (45) together determine the values of A_n and B_n.

4.3 Calculating the effect on the far-field acoustic pressure

We are now in a position to calculate the acoustic pressure. In order to keep the number of calculations within reasonable bounds, numerical results were obtained only for the far field where the azimuthal variation of p is independent of ξ. This can be seen by taking the limit as $\xi \to \infty$ of the right-hand side of equation (35) and using the principal asymptotic form of $H_n^{(2)}$ (described by result 9.2.4 of Ref 3) so as to obtain

$$p = p_i\sqrt{\frac{2}{\pi Ka\xi}}\, e^{i\{\omega t - Ka(\xi+\xi_0\cos\theta)+(\pi/4)\}} \times$$
$$\times \left\{1 + e^{iKa\xi_0\cos\theta}\sum_{n=-\infty}^{\infty}(-1)^n B_n H_n^{(2)}(Ka\xi_0)e^{in\theta} + 0\left(\frac{1}{Ka\xi}, \frac{\xi_0}{\xi}, \frac{Ka\xi_0^2}{\xi}\right)\right\} . \qquad \ldots\ldots (46)$$

The effect of the solid body rotation on the far-field azimuthal variation of p was assessed by evaluating the ratio of the amplitudes of p and p_i, *ie* the far-field directivity defined by

$$10\log_{10}\left|\frac{p}{p_i}\right| \doteq 10\log_{10}\left|1 + e^{iKa\xi_0\cos\theta}\sum_{n=-\infty}^{\infty}(-i)^n B_n H_n^{(2)}(Ka\xi_0)e^{in\theta}\right| . \qquad \ldots\ldots (47)$$

Eliminating A_n from equations (36) and (45) gives

$$B_n = \frac{\beta_n J_n(Ka) - \gamma_n J_n'(Ka)}{\gamma_n H_n^{(2)\prime}(Ka) - \beta_n H_n^{(2)}(Ka)} , \qquad (48)$$

where
$$\beta_n \equiv Ka\left\{(Ka + Mn)\alpha_n J_n'(\alpha_n) + 2MnJ_n(\alpha_n)\right\} \qquad (49)$$
and
$$\gamma_n \equiv (Ka + Mn)\left\{(Ka + Mn)^2 - 4M^2\right\}J_n(\alpha_n) . \qquad (50)$$

The infinite series in equation (47) has similar convergence properties to that which would be obtained if the rotating region were replaced by a hard circular cylinder. For, as $n \to \pm\infty$, $\gamma_n/\beta_n \to \infty$ and

$$(-i)^n B_n H_n^{(2)}(Ka\xi_0)e^{in\theta} \to -(-i)^n \frac{J_n'(Ka)}{H_n^{(2)\prime}(Ka)} H_n^{(2)}(Ka\xi_0)e^{in\theta} \qquad (51)$$

which is the exact form of the nth term in the series representing the far-field acoustic pressure scattered when $\frac{\partial p}{\partial r} = 0$ on $r = a$. Since the latter series is known to converge, it follows that the corresponding series in equation (47) also converges. There is a further similarity between the two series in that, to obtain a required accuracy, progressively more and more terms have to be summed as Ka increases.

The series in equation (47) was approximated by the partial sum from $n = -m$ to $+m$, where $m\ (>0)$ was chosen so that the terms omitted were smaller in magnitude than the rounding error of the computer upon which the calculations were performed. This procedure was justified by increasing m and showing that the effect was indeed negligible to the order of accuracy of the computer in single-precision arithmetic. The results are discussed in section 6.

5 THE EVALUATION OF THE ACOUSTIC PRESSURE AS AN ASYMPTOTIC POWER SERIES IN MACH NUMBER

5.1 Application of the method to vortex flow in general

We now return to equations (20) and (21), and solve them, when $M \ll 1$, by the second theoretical approach outlined in the introduction. Let $F(r) \equiv V/V_0$ where $V_0 \equiv \max|V|$, and let C_0 be a characteristic value of the speed of sound. The definitions of V_0 and C_0 are compatible with those adopted in section 3 provided $F = \frac{r}{a}$ for the solid body rotation. Using these more general definitions in equations (20) and (21), we obtain

$$\left(\frac{C_0^2}{C^2}\Delta^2 - \frac{\partial^2}{r^2\partial\theta^2} - \frac{\partial^2}{\partial z^2}\right)\frac{p}{R} + \left\{\Delta\left(\frac{\partial}{\partial r} + \frac{1}{r} + M^2\frac{C_0^2F^2}{C^2 r}\right) - M\frac{1}{r}\frac{drF}{dr}\frac{\partial}{r\partial\theta}\right\}C_0 u = 0 \ , \qquad \ldots\ldots (52)$$

and

$$\left(\Delta\frac{\partial}{\partial r} + 2M\frac{F}{r}\frac{\partial}{r\partial\theta}\right)\frac{p}{R} + \left(\Delta^2 + 2M^2\frac{F}{r^2}\frac{drF}{dr}\right)C_0 u = 0 \ , \qquad (53)$$

where $\Delta \equiv \frac{\partial}{C_0\partial t} + M\frac{F}{r}\frac{\partial}{\partial\theta}$. Equations (9) and (14) imply

$$\frac{dR}{dr}\Big/R = M^2\frac{C_0^2F^2}{C^2 r} \qquad (54)$$

and

$$\frac{dC}{dr}\Big/C = M^2\frac{R}{C}\left(\frac{\partial C}{\partial R}\right)_S\frac{C_0^2F^2}{C^2 r} \ , \qquad (55)$$

so that R and C are, formally at least, constant to $O(M)$.

In this method, it is assumed that p and u can be expressed as asymptotic power series in M, *ie*

$$p \sim \sum_{n=0}^{\infty} p_n M^n \qquad \text{and} \qquad u \sim \sum_{n=0}^{\infty} u_n M^n \quad , \tag{56}$$

where p_n and u_n are independent of M. Substituting these series into equations (52) and (53), noting equations (54) and (55), and equating powers of M, we find that the first two terms are given by

$$\left(\frac{\partial^2}{C_0^2 \partial t^2} - \frac{\partial^2}{r^2 \partial \theta^2} - \frac{\partial^2}{\partial z^2}\right) p_0 + \frac{\partial}{C_0 \partial t}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) R_0 C_0 u_0 = 0 \quad , \tag{57}$$

$$\frac{\partial^2 p_0}{C_0 \partial t \partial r} + \frac{\partial^2 R_0 C_0 u_0}{C_0^2 \partial t^2} = 0 \quad , \tag{58}$$

$$\left(\frac{\partial^2}{C_0^2 \partial t^2} - \frac{\partial^2}{r^2 \partial \theta^2} - \frac{\partial^2}{\partial z^2}\right) p_1 + 2\frac{F}{r}\frac{\partial^2 p_0}{C_0 \partial t \partial \theta} + \frac{\partial}{C_0 \partial t}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) R_0 C_0 u_1 + \left(\frac{F}{r}\frac{\partial}{\partial r} - \frac{1}{r}\frac{dF}{dr}\right)\frac{\partial R_0 C_0 u_0}{\partial \theta} = 0 \tag{59}$$

and

$$\frac{\partial^2 p_1}{C_0 \partial t \partial r} + \frac{F}{r}\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\frac{\partial p_0}{\partial \theta} + \frac{\partial^2 R_0 C_0 u_1}{C_0^2 \partial t^2} + 2\frac{F}{r}\frac{\partial^2 R_0 C_0 u_0}{C_0 \partial t \partial \theta} = 0 \quad , \tag{60}$$

where $R_0 \equiv \lim_{M \to 0} R$. The acoustic velocity terms can be eliminated from these equations to give

$$\left(\frac{\partial^2}{C_0^2 \partial t^2} - \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{r^2 \partial \theta^2} - \frac{\partial^2}{\partial z^2}\right) p_0 = 0 \quad , \tag{61}$$

and

$$\frac{\partial}{C_0 \partial t}\left(\frac{\partial^2}{C_0^2 \partial t^2} - \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{r^2 \partial \theta^2} - \frac{\partial^2}{\partial z^2}\right) p_1 = -2\left(F\frac{\partial^2}{C_0^2 \partial t^2} + r\frac{d}{dr}\frac{F}{r}\frac{\partial}{\partial r}\right)\frac{\partial p_0}{r \partial \theta} \quad . \tag{62}$$

The equation for p_0, which is the incident acoustic pressure, is the ordinary wave equation governing acoustic propagation in a stationary homogeneous medium, *ie* the basic state of the fluid when $M = 0$. The first order effect of the flow, in this model, is to introduce a source distribution (the right-hand 100

side of equation (62)) which is the result of interaction between the given incident acoustic pressure and that flow. The effect of the vortex can be expressed in a similar fashion to equation (47), that is

$$10 \log_{10}\left|\frac{p}{p_0}\right| = 10\,M \log_{10}e \times \mathrm{Re}\,\frac{p_1}{p_0} + O(M^2) \quad , \tag{63}$$

where Re denotes the real part.

5.2 The method applied to the problem of section 4

The bounded region of solid body rotation in the analysis of section 4 implies a non-dimensional azimuthal velocity component given by

$$F = \frac{r}{a} H(a - r) \quad , \tag{64}$$

where the Heaviside function $H(x)$ is defined by $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$; here it arises as a consequence of the vortex sheet at $r = a$. The incident acoustic pressure is given by equation (33), and so

$$p_0 = P_i e^{i\omega t} H_0^{(2)}\left(K\sqrt{r^2 + r_0^2 + 2rr_0 \cos\theta}\right) \quad , \tag{65}$$

which is a solution of equation (61). Substituting equation (64) into equation (62), we obtain

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{r^2\partial\theta^2} + K^2\right)p_1 = \frac{2i}{Ka}\left\{K^2 H(a - r) + \delta(r - a)\left(\frac{\partial}{\partial r} - \frac{1}{a}\right)\right\}\frac{\partial p_0}{\partial\theta} \quad , \tag{66}$$

where use has been made of the fact that p_1 is independent of z and has frequency ω . The factor $\delta(r - a)$, where δ is Dirac's delta function, arises from differentiating $H(a - r)$.

The solution of equation (66) can be found by convolving the source term with the two-dimensional free-field Green's function for Helmholtz's equation (appropriate to outgoing waves), namely

$$\frac{i}{4} H_0^{(2)}\left(K\sqrt{r^2 + r_i^2 - 2rr_i \cos(\theta - \theta_i)}\right) \quad .$$

Thus

$$p_1 = -\frac{P_i e^{i\omega t}}{2Ka}\frac{\partial}{\partial\theta}\int_{r_i=0}^{\infty}\int_{\theta_i=-\pi}^{\pi} r_i H_0^{(2)}\left(K\sqrt{r^2 + r_i^2 - 2rr_i\cos(\theta-\theta_i)}\right)\times$$

$$\times\left\{K^2 H(a-r_i) + \delta(r_i - a)\left(\frac{\partial}{\partial r_i} - \frac{1}{a}\right)\right\} H_0^{(2)}\left(K\sqrt{r_i^2 + r_0^2 + 2r_i r_0\cos\theta_i}\right) dr_i d\theta_i \quad ,$$

...... (67)

and if use is made of Graf's addition theorem as in equation (34), then

$$p_1 = -\frac{P_i e^{i\omega t}}{2Ka}\frac{\partial}{\partial\theta}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_{\theta_i=-\pi}^{\pi} e^{in(\theta-\theta_i)}e^{im(\theta_i-\pi)}d\theta_i \int_{r_i=0}^{\infty} r_i H_n^{(2)}(kr)J_n(Kr_i)\times$$

$$\times\left\{K^2 H(a-r_i) + \delta(r_i - a)\left(\frac{\partial}{\partial r_i} - \frac{1}{a}\right)\right\} H_m^{(2)}(Kr_0)J_m(Kr_i)dr_i d\theta_i \qquad (68)$$

$$= -\frac{\pi P_i e^{i\omega t}}{Ka}\frac{\partial}{\partial\theta}\sum_{n=-\infty}^{\infty} e^{in(\theta-\pi)}H_n^{(2)}(Kr)H_n^{(2)}(Kr_0)\times$$

$$\times\left[K^2\int_0^a r_i J_n^2(Kr_i)dr_i + J_n(Ka)\left\{KaJ_n'(Ka) - J_n(Ka)\right\}\right] \quad . \qquad (69)$$

The integration with respect to r_i can be performed using results 11.3.31, 9.1.5 and 9.1.27 of Ref 3 to give

$$p_1 = -\frac{i\pi P_i e^{i\omega t}}{Ka}\sum_{n=-\infty}^{\infty}(-1)^n n\times$$

$$\times\left\{\frac{K^2a^2 - n^2 - 2}{2}J_n^2(Ka) + KaJ_n(Ka)J_n'(Ka)\right.$$

$$\left. + \frac{K^2a^2}{2}J_n'^2(Ka)\right\} H_n^{(2)}(Kr_0)H_n^{(2)}(Kr)e^{in\theta} \quad .$$

...... (70)

Finally, using equation (63) and the principal asymptotic form of $H_n^{(2)}$, we can show that in the far field

$$10 \log_{10}\left|\frac{p}{p_0}\right| \doteq -\frac{10\pi M \log_{10}e}{Ka} \operatorname{Re} ie^{iKa\xi_0 \cos\theta} \sum_{n=-\infty}^{\infty} (-i)^n n \times$$

$$\times \left\{\frac{K^2a^2 - n^2 - 2}{2} J_n^2(Ka) + KaJ_n(Ka)J_n'(Ka) + \frac{K^2a^2}{2} J_n'^2(Ka)\right\} H_n^{(2)}(Ka\xi_0)e^{in\theta} \quad \text{...... (71)}$$

$$= \frac{10\pi M \log_{10}e}{Ka} \operatorname{Im} e^{iKa\xi_0 \cos\theta} \sum_{n=-\infty}^{\infty} (-i)^n n \times$$

$$\times \left\{\frac{K^2a^2 - n^2 - 2}{2} J_n^2(Ka) + KaJ_n(Ka)J_n'(Ka) + \frac{K^2a^2}{2} J_n'^2(Ka)\right\} H_n^{(2)}(Ka\xi_0)e^{in\theta} \quad , \quad \text{...... (72)}$$

where Im denotes the imaginary part.

There are a number of interchanges of limiting processes in going from equation (67) to (72). They can all be shown to be justified since the region of integration is effectively finite and the series are all uniformly convergent. It can also be shown that the form of the nth term in equation (72) is the same as that which would be obtained from equation (47) by writing it correct to $O(M)$. This provides a check on the working used to derive both equations.

The most interesting feature of the final expression is that it is an odd function of θ . This is a general property of $10 \log_{10}\left|\frac{p}{p_0}\right|$, correct to $O(M)$, when p_0 is an even function of θ . This can be deduced from equation (62), since its source term, which is a function of $\frac{\partial p_0}{\partial \theta}$, must be odd, and its operator on p_1 preserves parity. This feature has also been demonstrated by Broadbent[2], starting from the expressions obtained in the ray theory limit.

6 DESCRIPTION AND DISCUSSION OF RESULTS

6.1 The presentation of results

The main body of results presented in Figs 2 to 16 are calculated from equation (47). The values of the dependent variable are shown only for the range of θ from -45° to $+45^{\circ}$ where the vortex has the most significant effect on the acoustic pressure. In this particular example there is also a significant effect in the region on the opposite side of the solid body rotation because the vortex sheet at the edge of the rotation causes back scattering in a manner analogous to a solid circular cylinder. However, this region is ignored since the back scattering would not be present in more practical cases where the basic velocity field is continuous.

A further restriction on the results presented here is that they are all for a single value of ξ_0 ($\xi_0 = 2$) so as to study in detail the variations brought about by changes in Mach number and wave number. Some calculations, which were done for higher values of ξ_0, indicate that the directivity pattern, for fixed values of M and Ka, is reduced both in amplitude and angular spread as the distance of the source from the centre of rotation increases.

The graphs of far-field directivity are grouped in three different ways depending on the range of Mach number and wave number. Figs 2 to 7 are designed to show how the directivity changes as $M \to 0$ at fixed Ka. Since equation (72) shows that $\lim_{M \to 0} \frac{1}{M} \log\left|\frac{p}{p_0}\right|$ is independent of M, the two curves calculated from equation (47), at each fixed wave number, are scaled with Mach number so that they should be close to the third curve which is calculated from equation (72) for a Mach number of 0.1. The latter value of M is chosen for convenience, since then $\frac{1}{M} \log_{10}\left|\frac{p}{p_0}\right|$ equals the far-field directivity defined by equation (47).

Figs 8 to 10 and 11 to 16 both have the same dependent variable calculated from equation (47), but the former group emphasises dependence of far-field directivity on wave number whereas the latter group compares directivities at fixed Ka, again, for the higher Mach numbers. At these values of M, the low Mach number approximation made in section 3 is probably beginning to break down.

6.2 The variation of far-field directivity with Mach number and wave number

The dependence of far-field directivity on Mach number and wave number is shown to be quite complicated. However, there are some trends which can be identified from Figs 2 to 16. As $M \to 0$ with Ka fixed, the directivity tends to the odd function of θ predicted by the analysis of section 5 correct to $O(M)$ 100

but, as M increases away from zero, the directivity becomes less like an odd function. For the lower values of Ka, for which the directivity pattern remains relatively simple, this divergence may be characterised by the behaviour of the three parameters θ_0, I_{max} and I_{min} which are defined respectively by the smallest positive value of θ where $10 \log_{10}\left|\frac{p}{p_i}\right|$ is zero, the maximum value of $10 \log_{10}\left|\frac{p}{p_i}\right|$, and the minimum value of $10 \log_{10}\left|\frac{p}{p_i}\right|$. These parameters are plotted against Mach number, for $Ka = 5$, 10 and 15, in Fig 17. This shows that as M increases, θ_0 increases away from the value zero predicted by the $O(M)$ theory of section 5, and that I_{max} is increasingly overpredicted and $|I_{min}|$ increasingly underpredicted by the $O(M)$ theory.

For all the values of M and Ka covered by Fig 17, the directivity can be simply described as giving a region of decreased acoustic pressure for $\theta < \theta_0$ followed by a region of increased acoustic pressure for $\theta > \theta_0$. The value of M, upto which this simple picture holds, decreases as Ka increases and the directivity evolves into an interference pattern which is probably better described by ray theory. The numbers of maxima and minima, in the region $-30^o \leqslant \theta^o \leqslant 30^o$, are plotted against wave number at fixed Mach number in Fig 18. This clearly shows that as Ka increases, the numbers increase and thus the angular distance between directions of constructive and destructive interference decreases, an important property of interference. It is hoped to give a detailed explanation of how this interference pattern arises, as well as an assessment of the accuracy of ray theory in this example, in a future publication.

The overall implication of these trends is that the asymptotic expansion in powers of M is not uniformly valid as $Ka \to \infty$. The reason for this can be readily seen from the remarks made on the convergence of the infinite series in equation (47). As Ka increases, more terms have to be included to achieve a given accuracy. This implies that the average value of $|Mn|$, amongst the significant terms, increases and it reaches a point where the effect of this parameter can no longer be regarded as small enough for the asymptotic power series approach to be adequate.

More generally, ray theory shows that it is necessary to distinguish between the phase and amplitude of the acoustic pressure. Thus an obvious improvement to the method of section 5 is to adopt an asymptotic expansion in powers of M for both amplitude and phase when $Ka \gg 1$.

6.3 Assessment of the utility of the asymptotic power series in M for acoustic pressure

In section 5, the asymptotic expansion was taken only to $O(M)$. In the example used in the present Report however, that order of accuracy was achieved with about the same amount of computational effort as was required for the inherently more accurate method of analysis described in sections 3 and 4. To take the asymptotic expansion a stage further to $O(M^2)$ would require at least twice the effort needed for the other approach. Although this may not be a generally applicable criticism, in that the functions derived in section 3 are particularly simple for the solid body rotation, the results have shown that the asymptotic expansion is not uniformly valid as $Ka \to \infty$, and so this method will not give necessarily useful estimates of the vortex refraction effect in practical cases.

It seems to the author that the best approach to the general problem of predicting the effect of a vortex, and indeed the propagation effects of all low Mach number flows, is to retain the $O(M)$ terms in the operator on acoustic pressure and then attempt to solve the resulting approximate acoustic equation as accurately as possible. An example of the successful application of this approach, in the case of potential flow, is described in Ref 5.

7 CONCLUSIONS

The scattering of sound by a region of fluid in solid body rotation at low Mach number is an easily analysed example which can be used as a test case for assessing the accuracy of various approximate methods for calculating the effect on acoustic propagation of more practical vortex flows. The main body of results presented in this Report are obtained by solving the acoustic equation after ignoring certain terms which are $O(M^2)$ and are believed to have a uniformly small effect at least up to the maximum Mach number considered of 0.3. For this reason, the method is considered to be suitable for use as a check against which other approximate methods can be tested.

One such method is that of an asymptotic expansion in powers of M. It has been shown that such an expansion is not uniformly valid for large Ka and, when tested against the uniformly valid results, it is found to be seriously in error for $Ka = 15$ even at a Mach number of 0.1.

For large wave numbers the directivity of acoustic pressure takes on the appearance of an interference pattern. This implies that ray theory will provide a better description at these wave numbers and the next task will be to assess the accuracy of this approximation. 100

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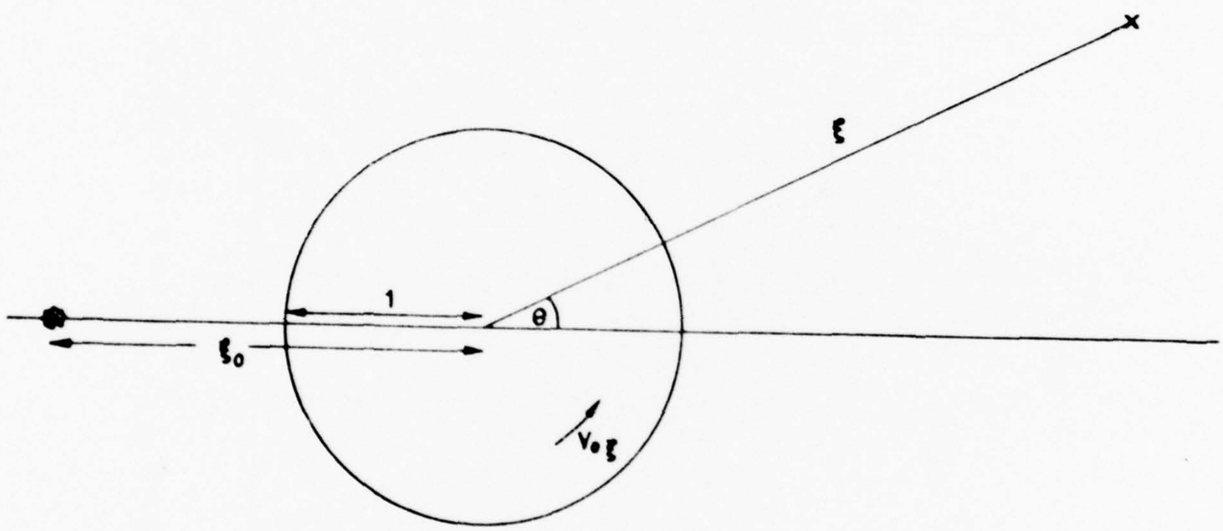

Fig 1 Illustration of the problem described in section 4.1

Fig 2

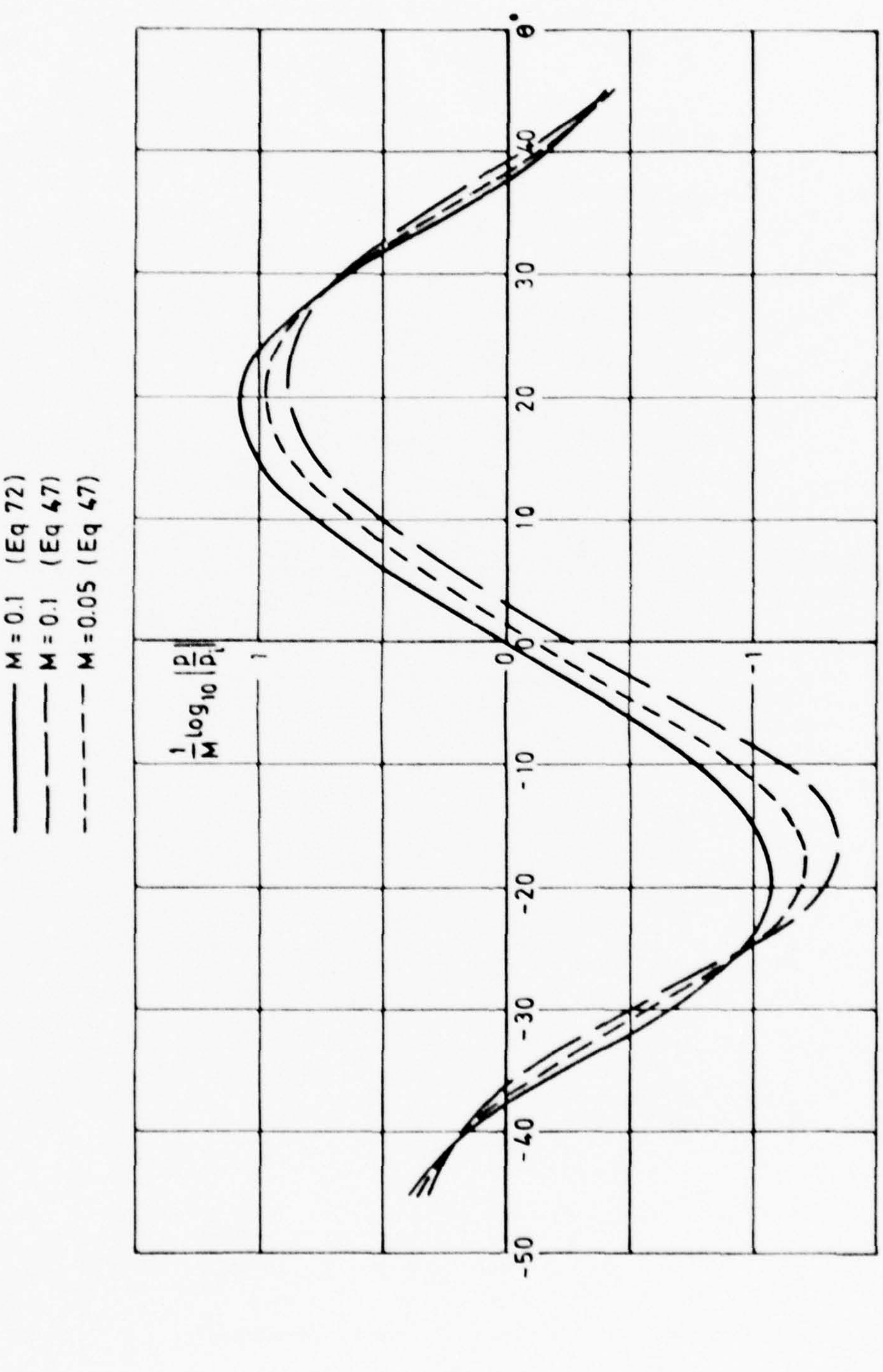

Fig 2 The behaviour of $\frac{1}{M}\log_{10}\left|\frac{p}{p_i}\right|$ as $M \to 0$ for $Ka = 5$

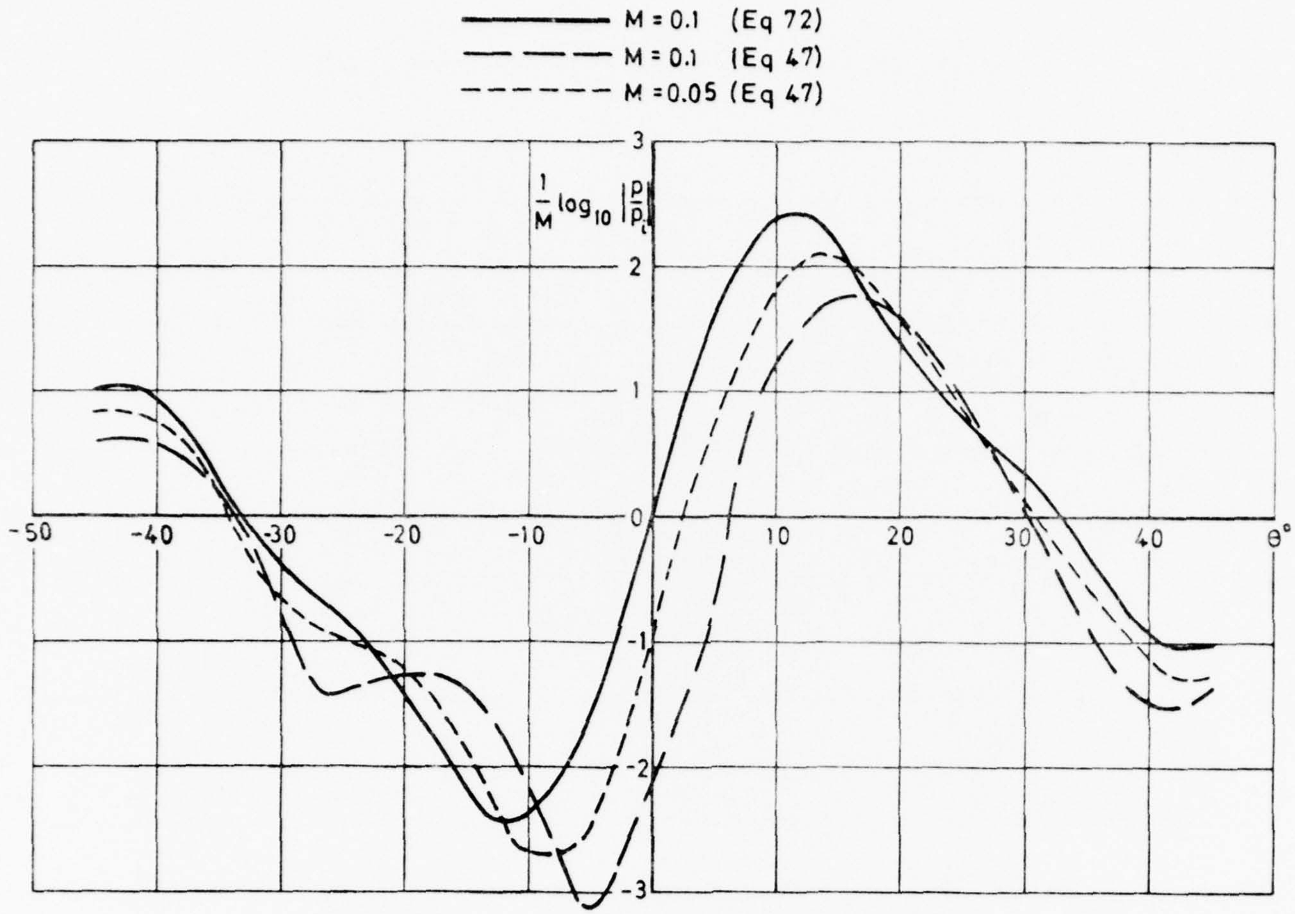

Fig 3 The behaviour of $\frac{1}{M}\log_{10}\left|\frac{p}{p_i}\right|$ as $M \to 0$ for $Ka = 10$

Fig 4

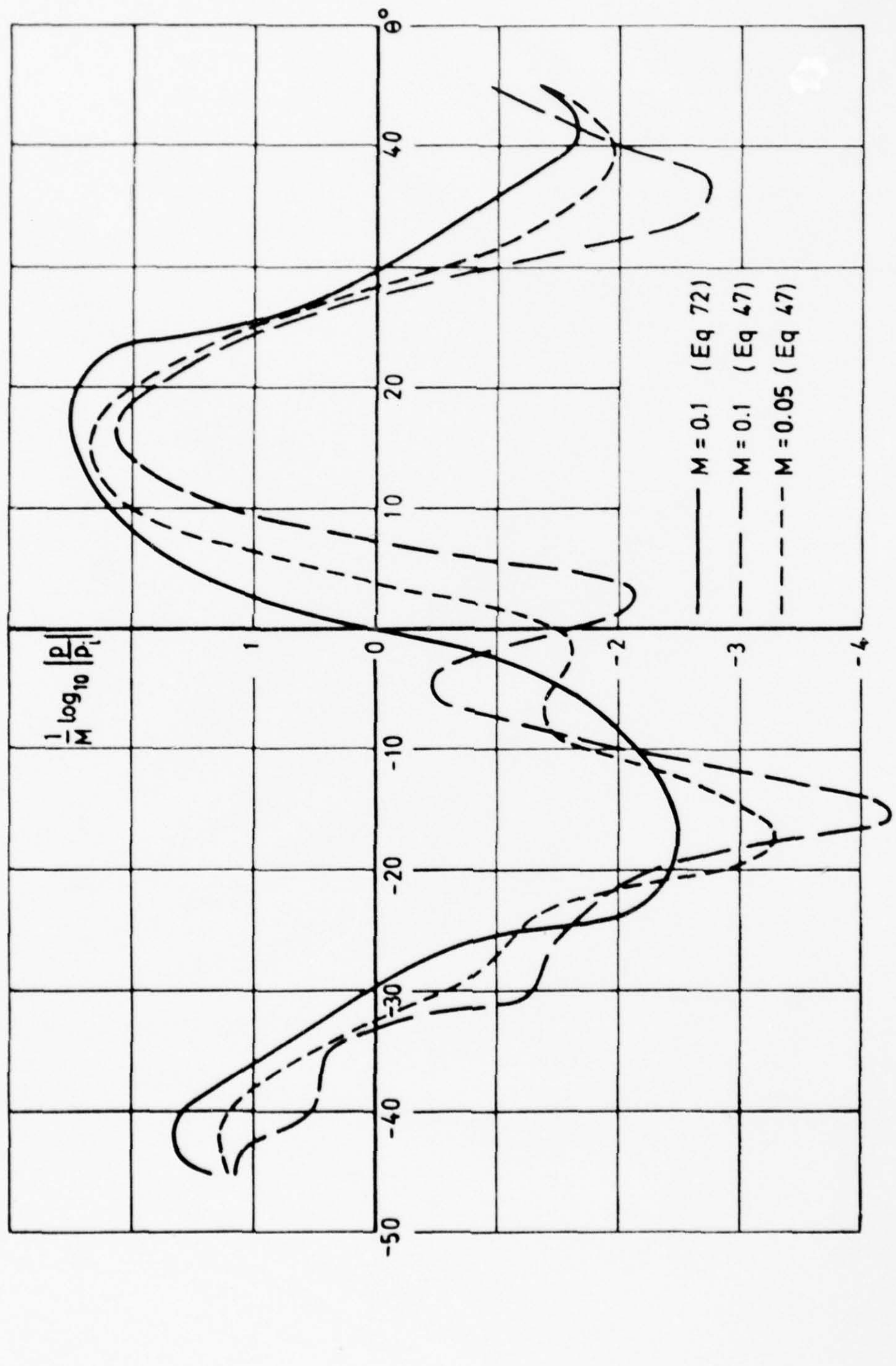

Fig 4 The behaviour of $\frac{1}{M}\log_{10}\left|\frac{p}{p_i}\right|$ as $M \to 0$ for $Ka = 15$

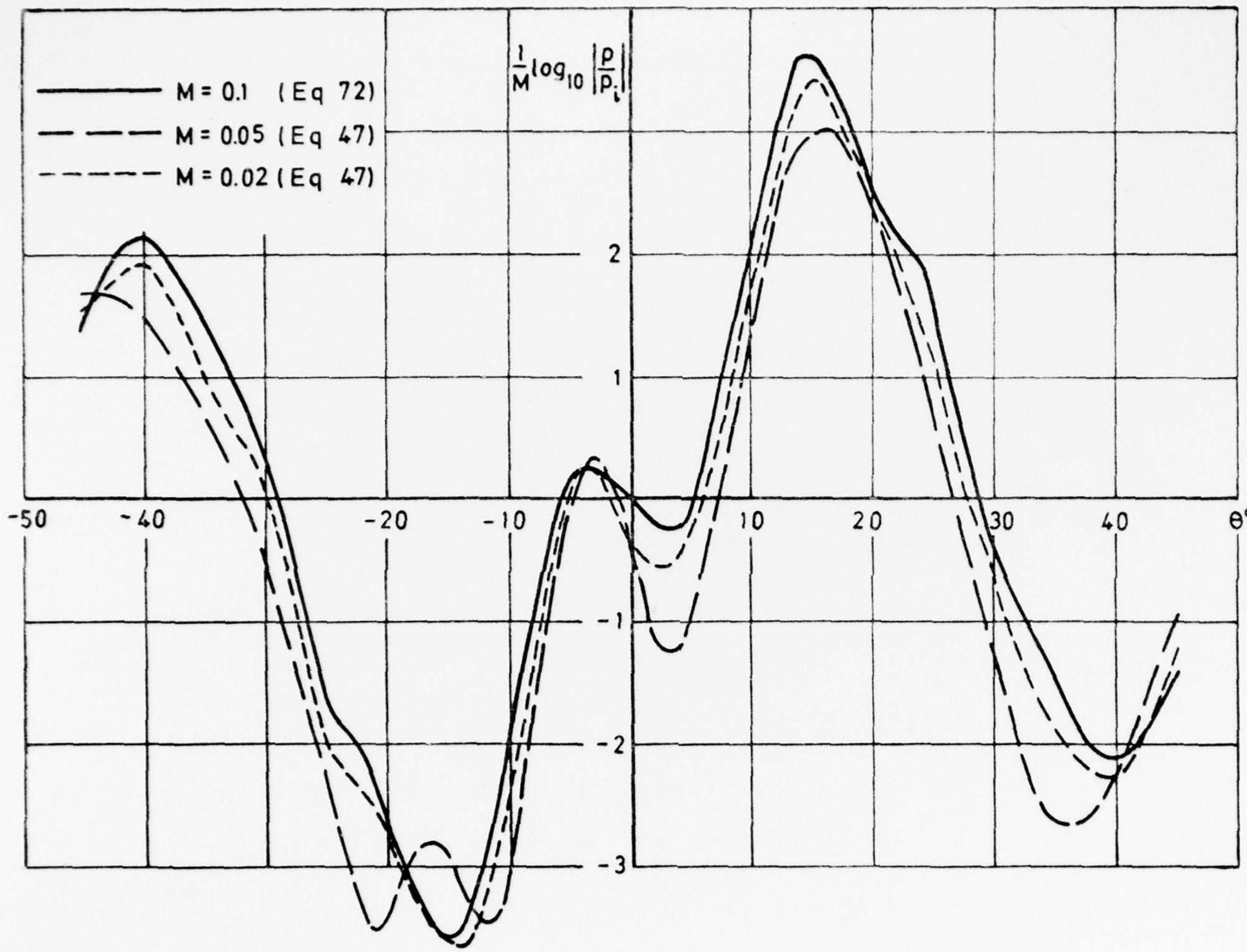

Fig 5 The behaviour of $\frac{1}{M}\log_{10}\left|\frac{p}{p_i}\right|$ as $M \to 0$ for $Ka = 20$

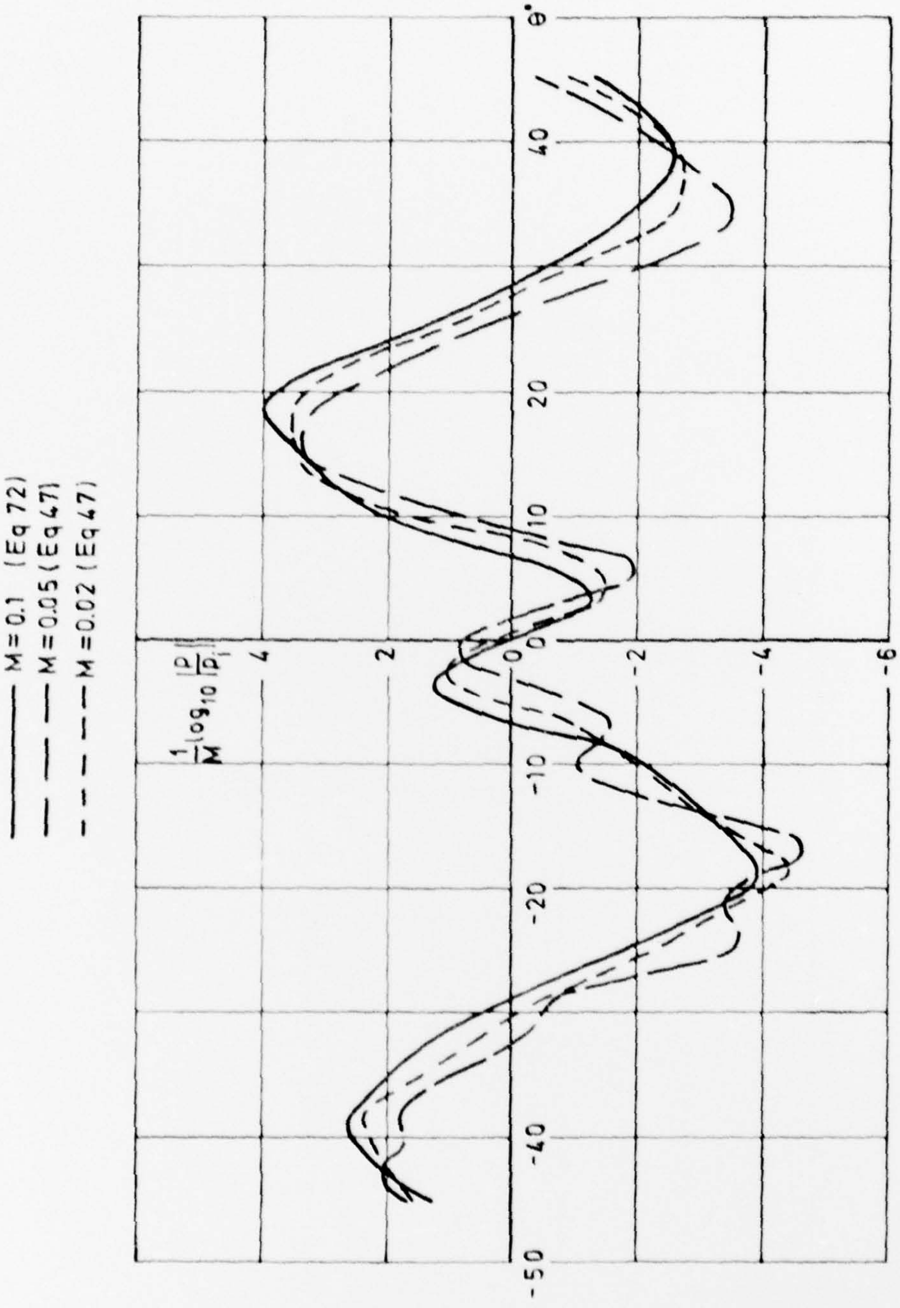

Fig 6 The behaviour of $\frac{1}{M}\log_{10}\left|\frac{p}{p_i}\right|$ as $M \to 0$ for $Ka = 25$

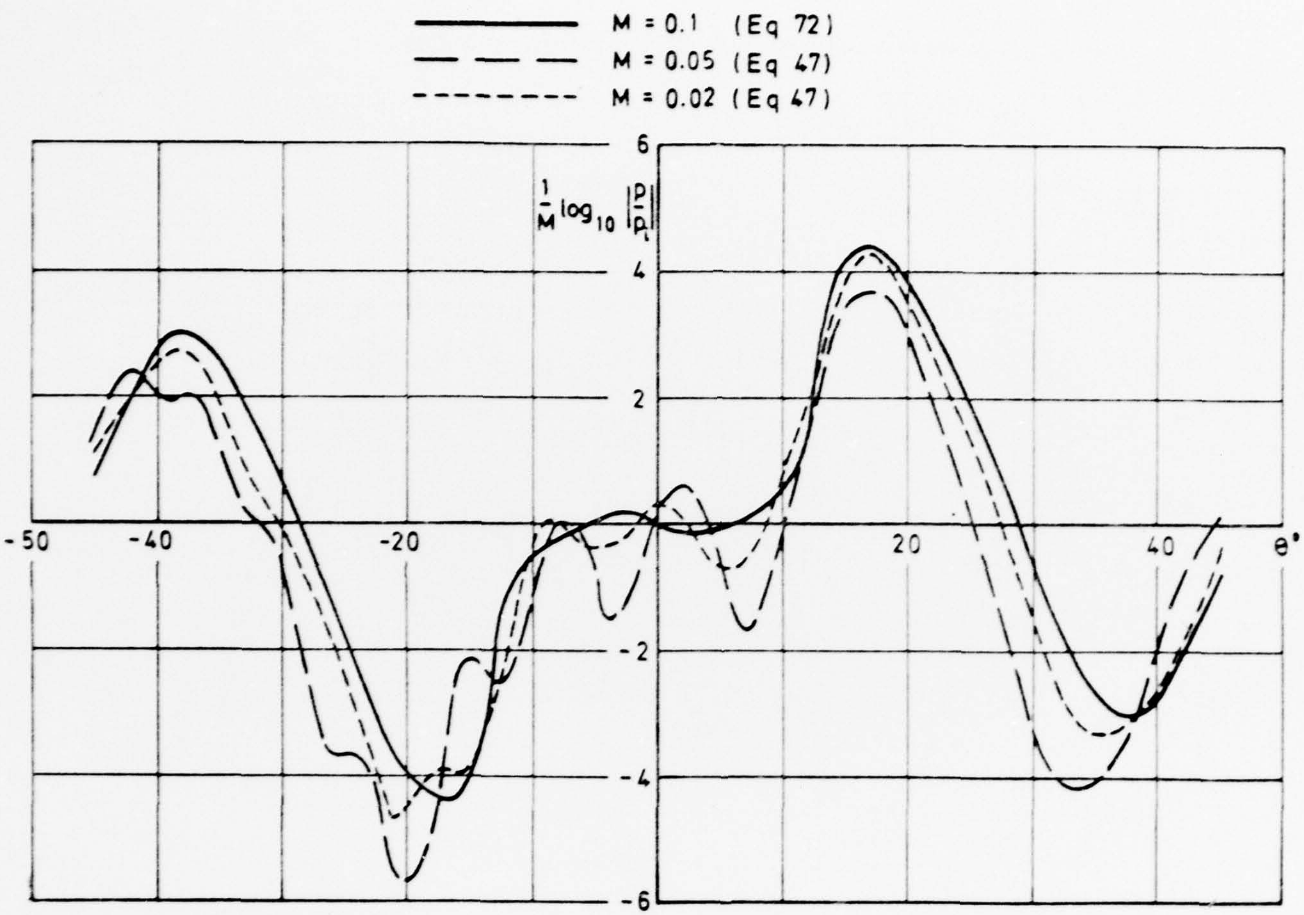

Fig 7 The behaviour of $\frac{1}{M}\log_{10}\left|\frac{p}{p_i}\right|$ as $M \rightarrow 0$ for $Ka = 30$

Fig 7

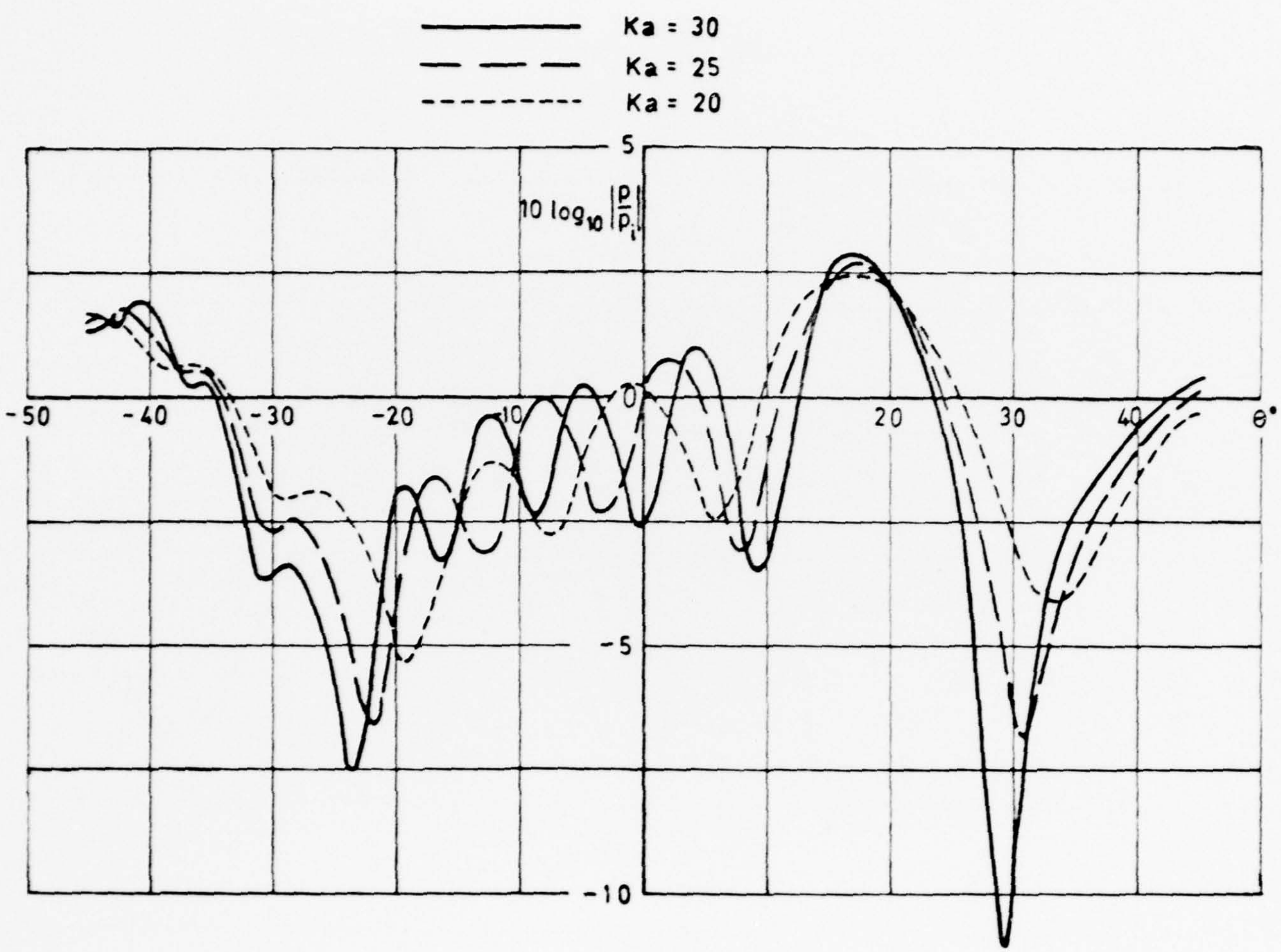

Fig 8 The effect of wave number on far-field directivity for M = 0.1

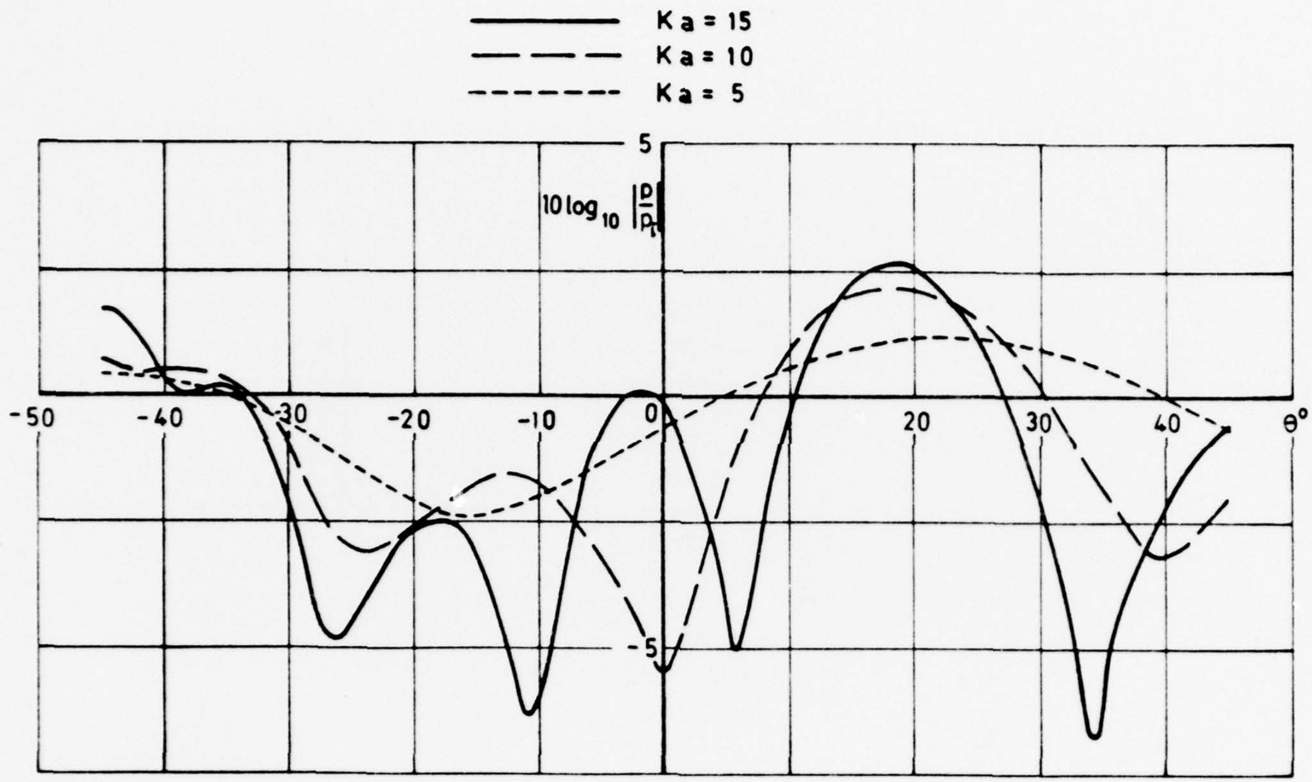

Fig 9 The effect of wave number on far-field directivity for M = 0.15

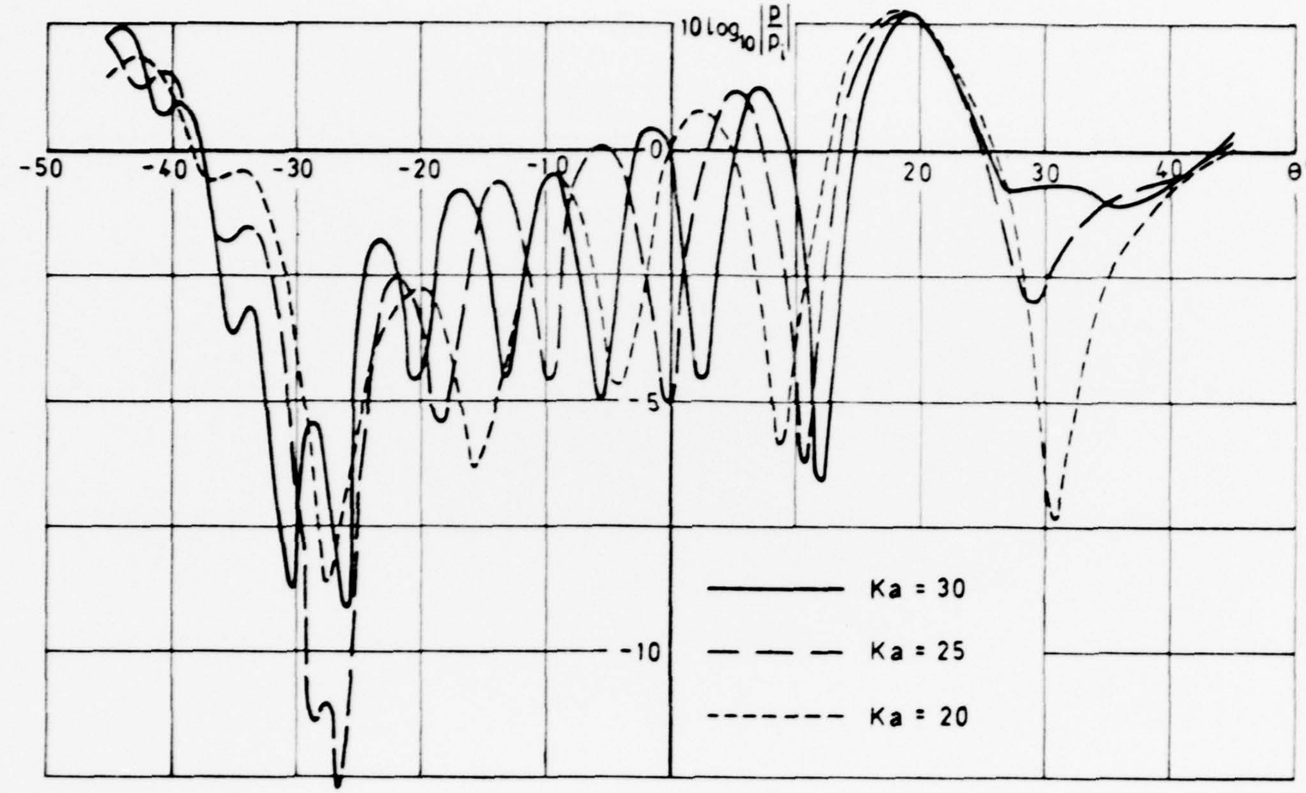

Fig 10 The effect of wave number on far-field directivity for M = 0.15

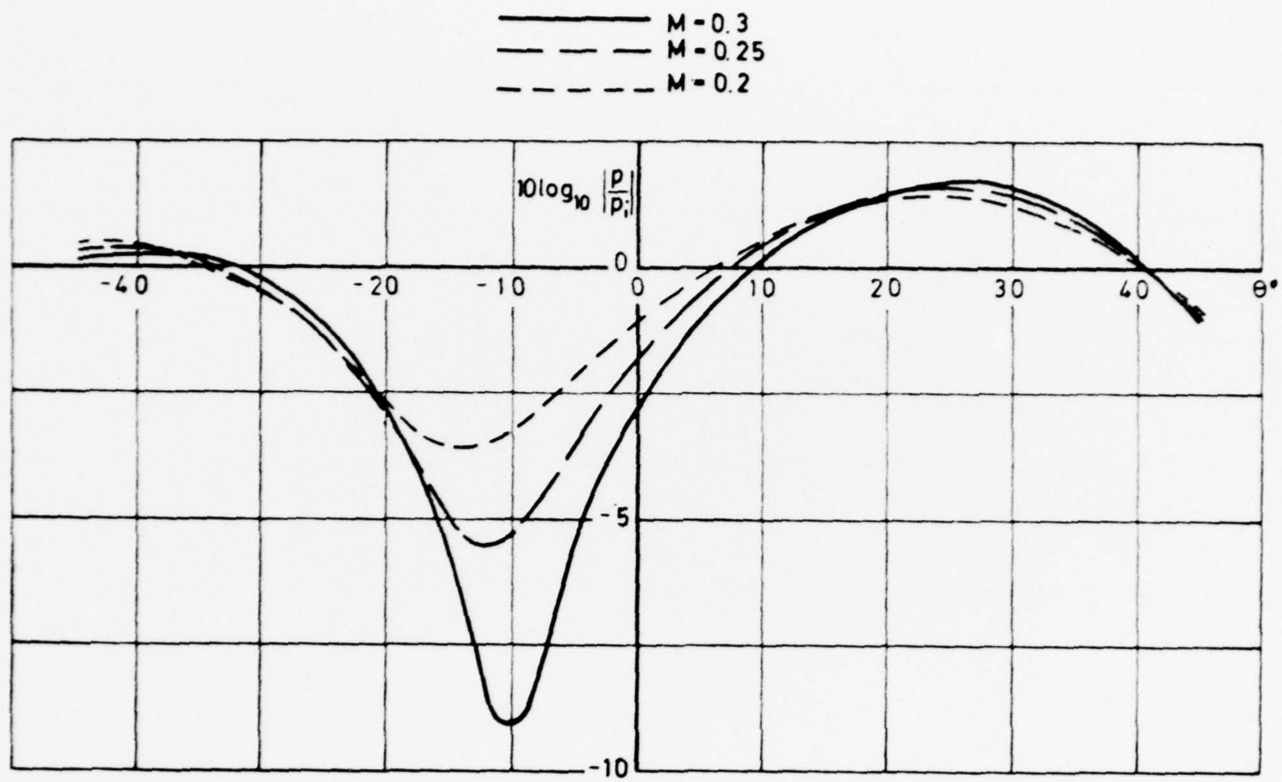

Fig 11 The effect of Mach number on far-field directivity for Ka = 5

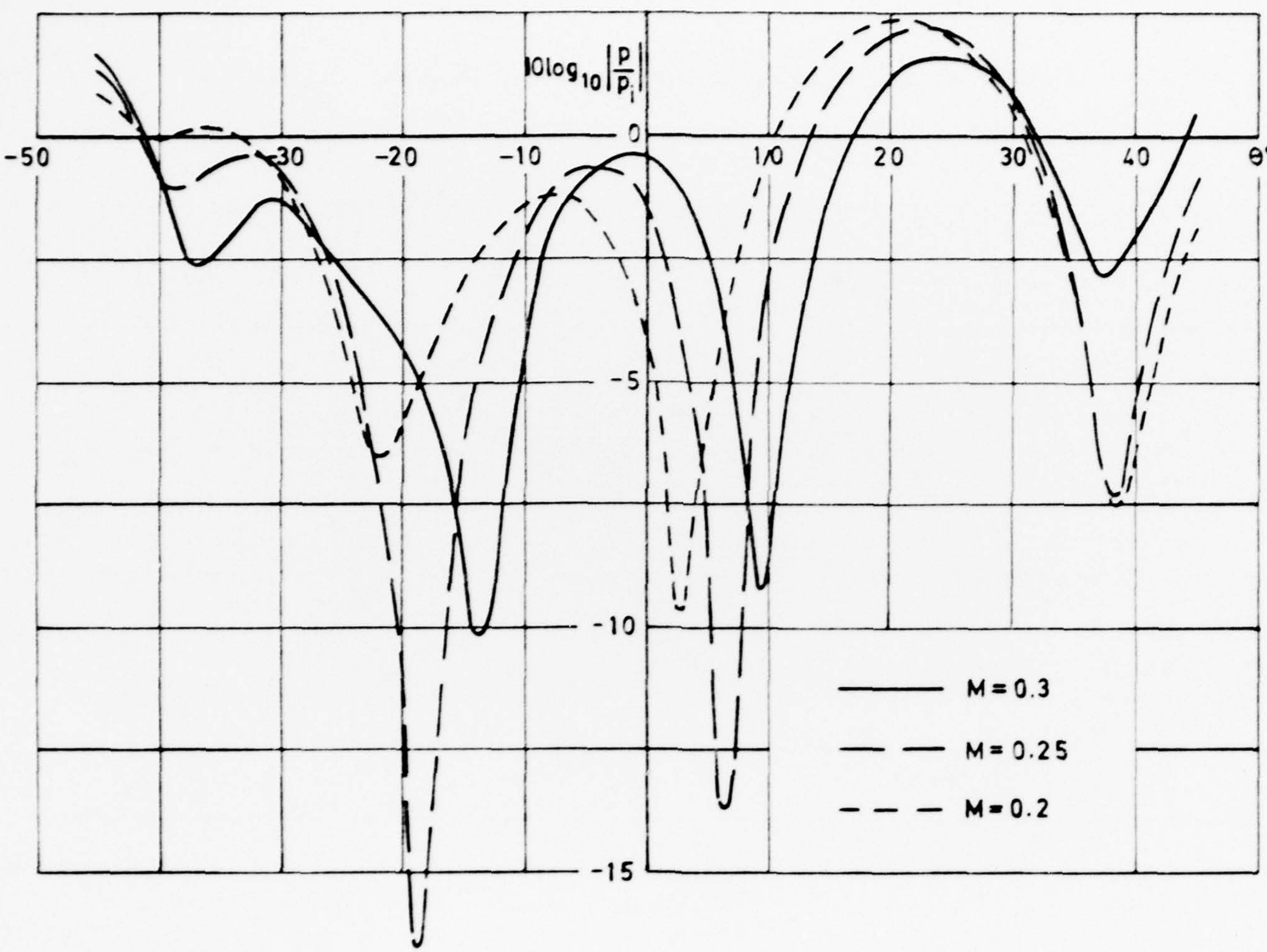

Fig 12 The effect of Mach number on far-field directivity for Ka = 10

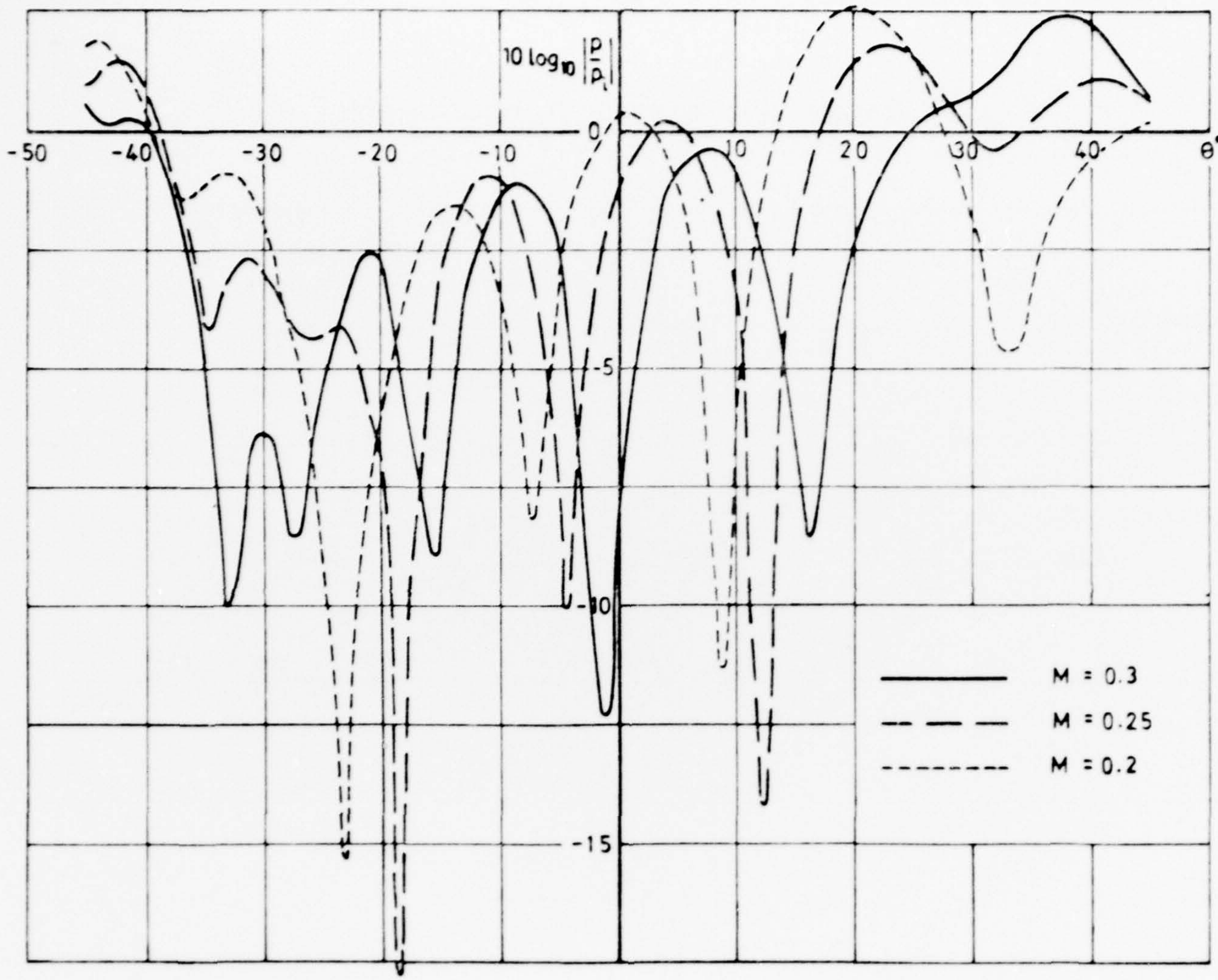

Fig 13 The effect of Mach number on far-field directivity for Ka = 15

Fig 14

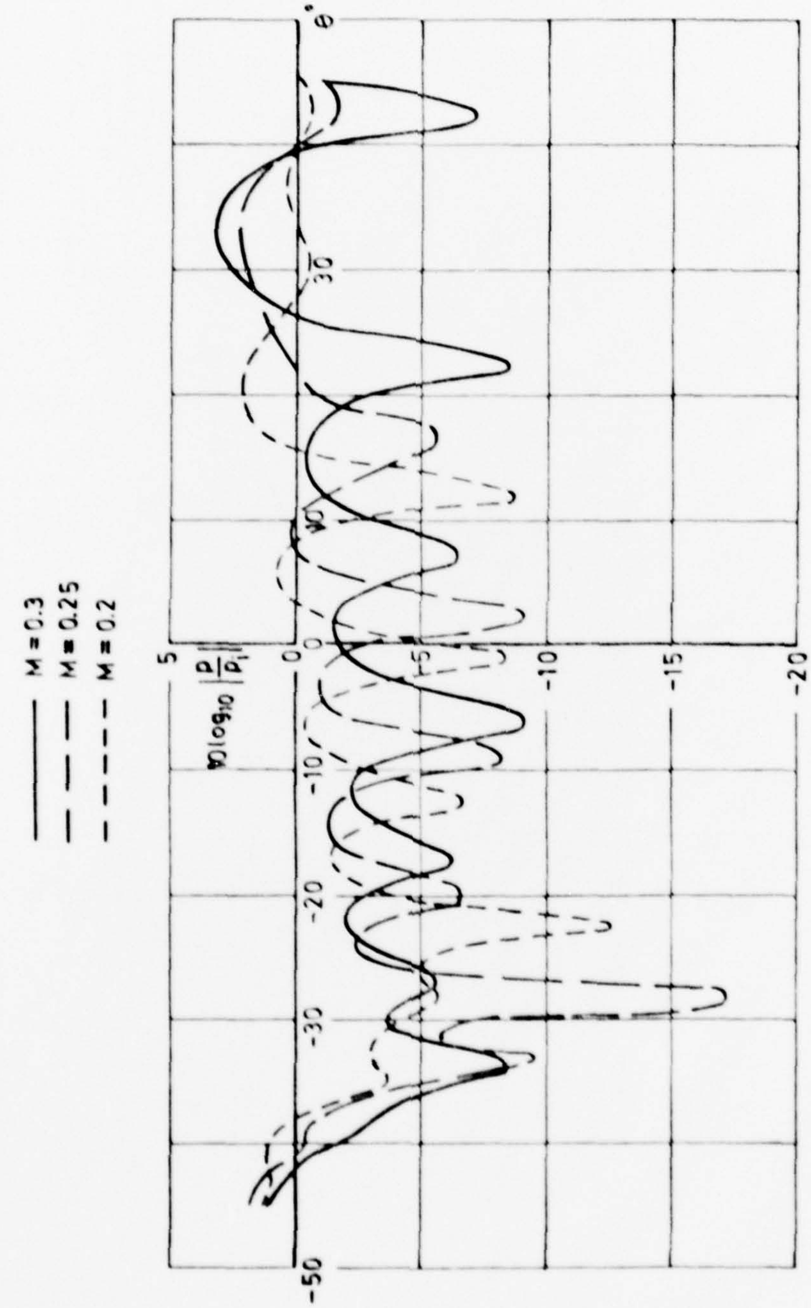

Fig 14 The effect of Mach number on far-field directivity for Ka = 20

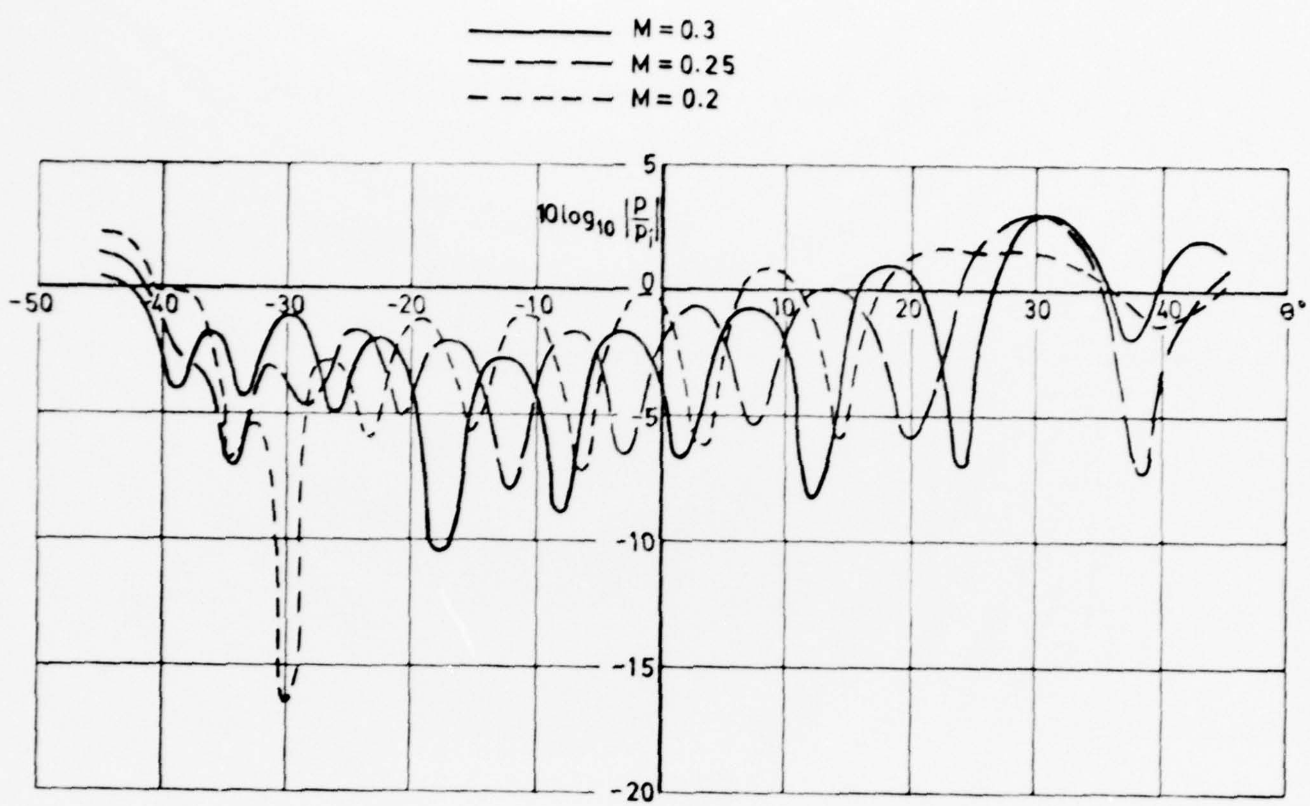

Fig 15 The effect of Mach number on far-field directivity for Ka = 25

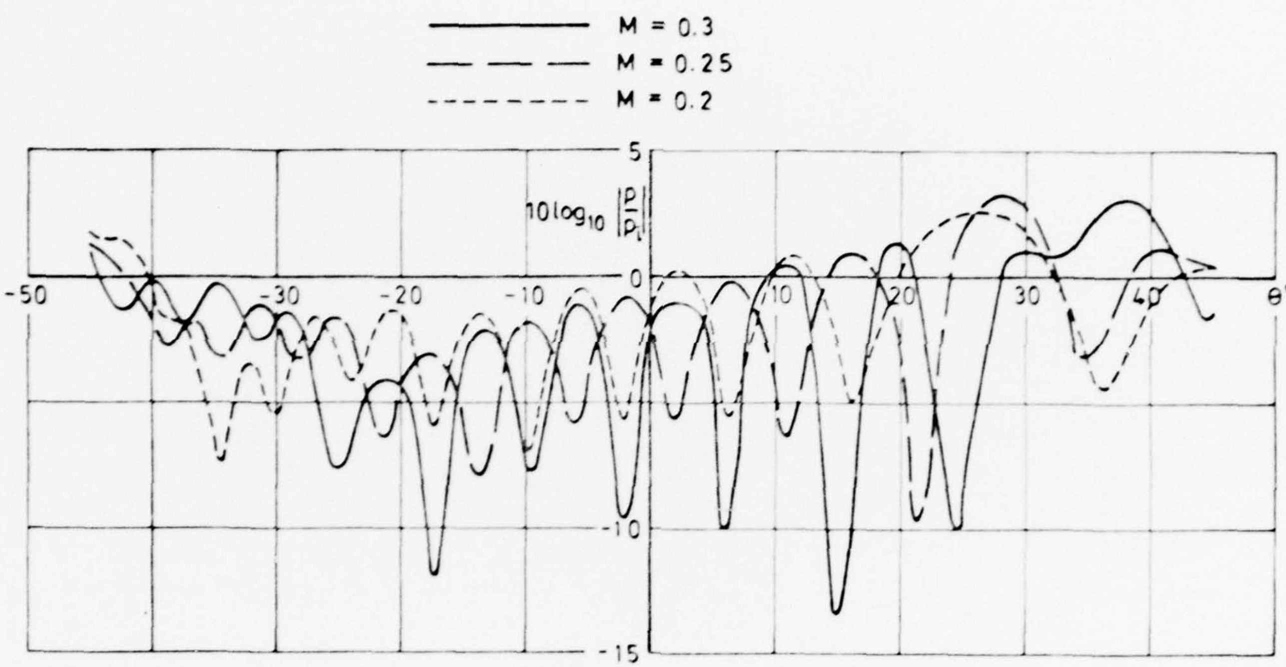

Fig 16 The effect of Mach number on far-field directivity for Ka = 30

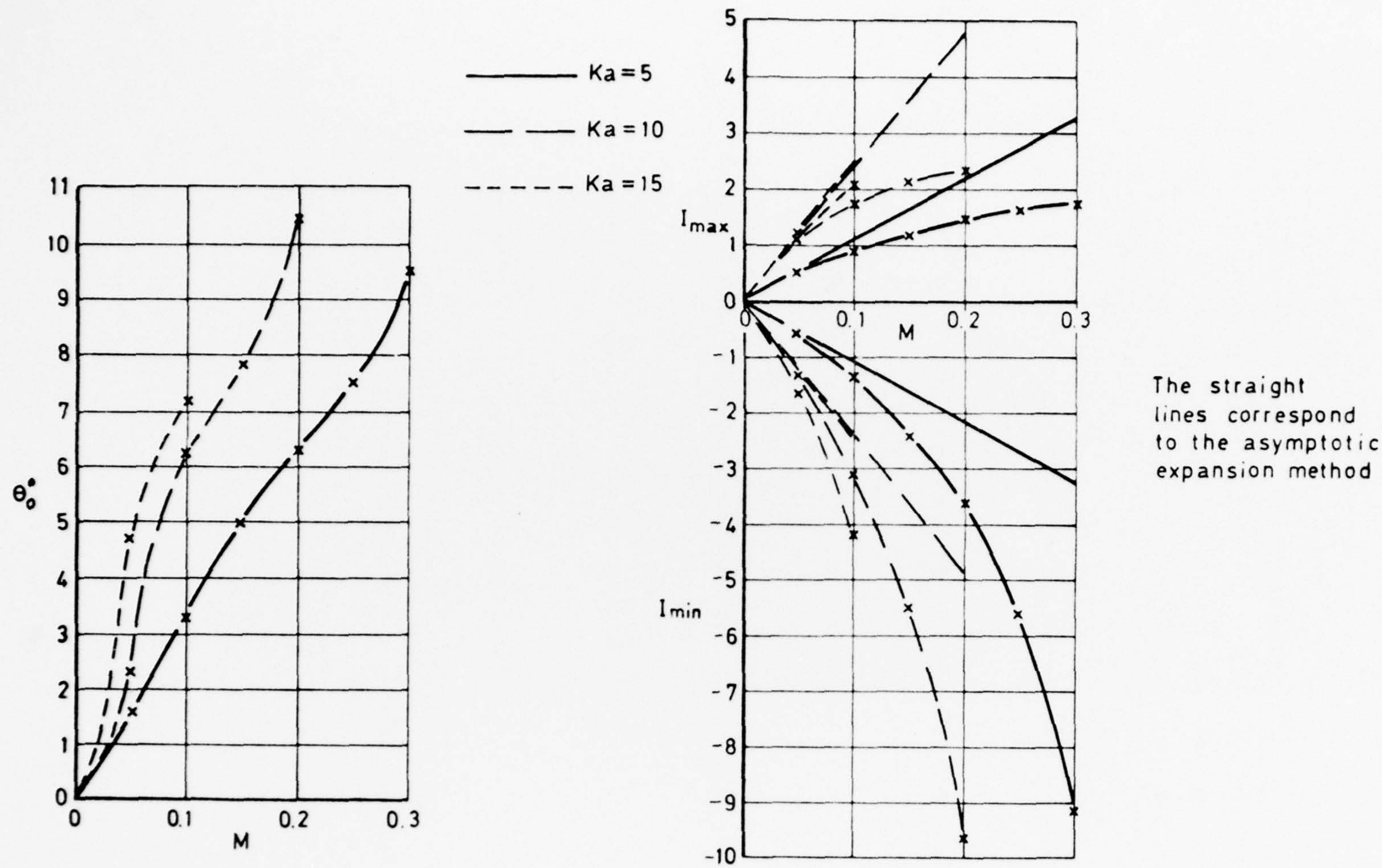

Fig 17 Divergence of directivity from the result correct to O(M) obtained by the asymptotic expansion method

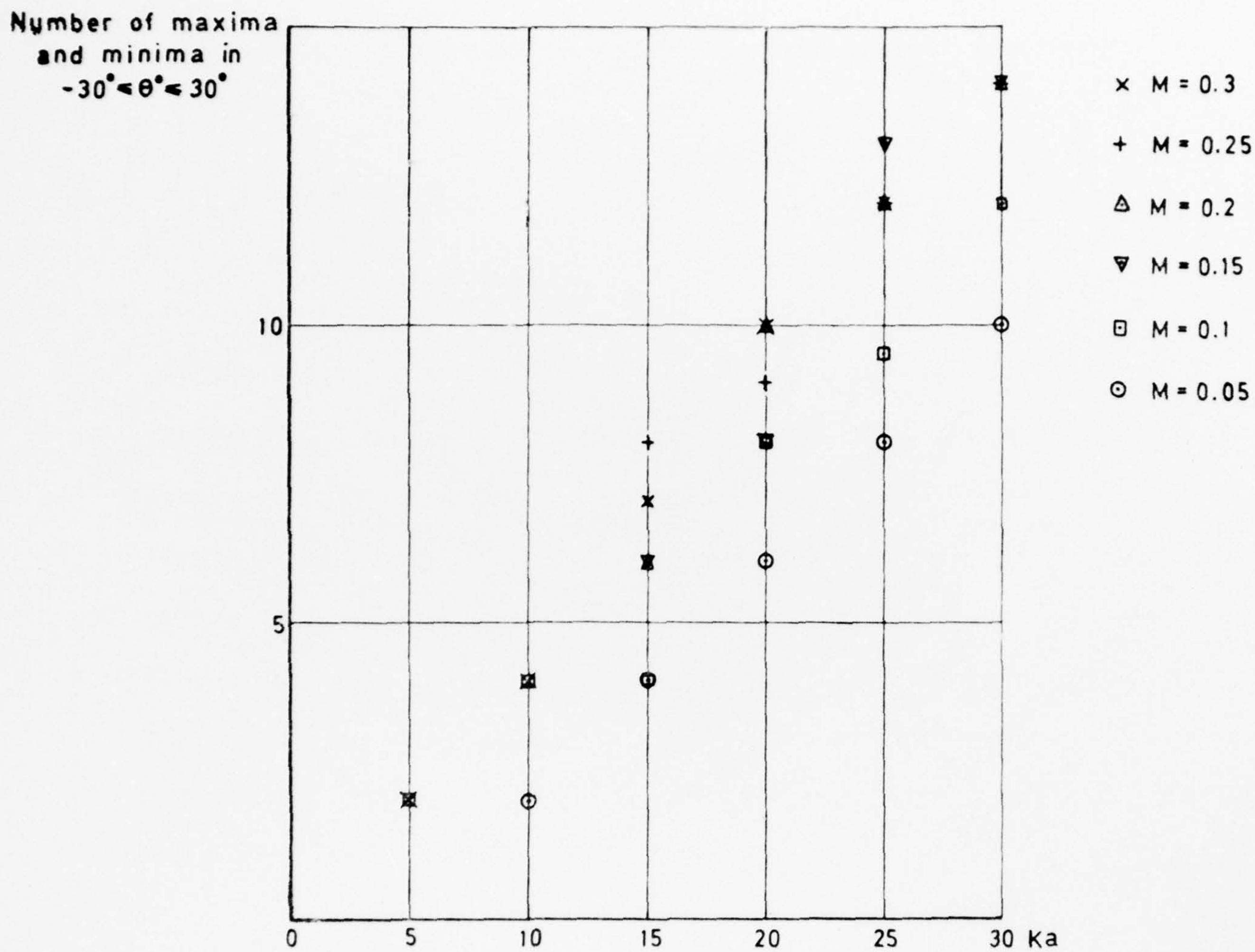

Fig 18 Variation in the density of maxima and minima in the directivities

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1. DRIC Reference (to be added by DRIC)	2. Originator's Reference	3. Agency Reference	4. Report Security Classification/Marking
	RAE TR 78100	N/A	UNCLASSIFIED

5. DRIC Code for Originator	6. Originator (Corporate Author) Name and Location
850100	Royal Aircraft Establishment, Farnborough, Hants, UK
5a. Sponsoring Agency's Code	**6a. Sponsoring Agency (Contract Authority) Name and Location**
N/A	N/A

7. Title The scattering of sound by a region of fluid in solid body rotation at low Mach number

7a. (For Translations) Title in Foreign Language

7b. (For Conference Papers) Title, Place and Date of Conference

8. Author 1. Surname, Initials	9a. Author 2	9b. Authors 3, 4	10. Date	Pages	Refs.
Taylor, K.	-	-	August 1978	39	6

11. Contract Number	12. Period	13. Project	14. Other Reference Nos.
N/A	N/A		Aero 3440

15. Distribution statement
(a) Controlled by –
(b) Special limitations (if any) – Unlimited

16. Descriptors (Keywords) (Descriptors marked * are selected from TEST)

Acoustics. Scattering. Noise. Vortex refraction.

17. Abstract

This example, which is one of the simplest to exhibit vortex refraction of sound, is solved using an approximation which is uniformly valid at all frequencies and small Mach numbers. The region of solid body rotation is bounded by a vortex sheet which implies that the results obtained have little physical relevance. However this flow field gives acoustic equations which are comparatively easy to solve and thus an ideal test case for the various approximate techniques used to investigate the effect of more realistic vortices on sound.

The present Report gives an assessment of a method which sets out to express the acoustic pressure as an asymptotic power series in the Mach number. It is found that this approach produces an error which increases progressively as the frequency increases at a fixed Mach number. At higher frequencies it appears that the ray theory approximation is likely to be more accurate and it is hoped to publish an assessment of the utility of this method later.

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